# A New Method for Statistical Metric Multidimensional Unfolding

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*Abstract*

Metric multidimensional unfolding is a statistical estimation problem where the data structure is a set of measures that are monotonic functions of Euclidean distances between a number of observers and targets in a multidimensional space. The new method presented in this paper deals with estimating the target locations and the observer positions when the observations are functions of the squared distances between observers and targets observed with an additive random error in a two dimensional space. The method is based on the work of Cahoon (1976), Cahoon, Hinich and Ordeshook (1978) and Hinich (1978). The Cahoon-Hinich (C-H) method is a statistical metric multidimensional unfolding method that is based on the multidimensional spatial theory of electoral competition originally developed by Davis and Hinich (1966). The main result in this paper is a significant modification of the Cahoon-Hinich method that yields much robust estimates of the target locations in a two dimensional space for the parametric structure of the data generating model presented in the paper. The modification also yields more accurate estimates of the mean and variances of the observer locations than the original method. The data is transformed so that the nonlinearity due to the squared observer locations is removed. The sampling properties of the estimates are derived from the asymptotic variances of the additive errors of a maximum likelihood factor analysis of the sample covariance matrix of the transformed data augmented with bootstrapping. The robustness of the new method is tested using artificial data. The method is applied to a 2001 survey data set from Turkey to provide a real data example.

Keywords: Spatial theory, metric multidimensional unfolding, maximum likelihood factor analysis, least squares

## *1. INTRODUCTION*

The various multidimensional unfolding techniques that have been mainly developed by measurement psychologists originated in the work by Coombs (1964). Unfolding theory uses a geometric model for preferences and choice that posits that an individual will choice the alternative in the a multidimensional choice set that is closest to that person's ideal point in the space (De Leeuw, 2005). If the distance metric is either Euclidean distance or squared Euclidean distance then the unfolding theory is identical to the model of political choice introduced by Davis and Hinich (1966) in their theory of political competition.

Metric multidimensional unfolding is a statistical estimation problem where the data structure is a set of measures that are monotonic functions of Euclidean distances between a number of observers located at positions $\mathbf{x}_i$ and targets at locations $\boldsymbol{\pi}_m$. The first approach to locating the targets and the observers is given by Schonemann (1970).

Metric multidimensional unfolding is related to metric multidimensional scaling methods but the scaling methods developed from the approach originated by Torgerson (1952, 1958) is based on measures of distance between the targets as reported by the observers rather than the distances to the targets. Multidimensional scaling and unfolding has been applied in marketing, anthropology, psychology, and sociology (Weller and Romney, 1990), political science (Poole and Rosenthal, 1984, 1991, 1997) and Poole (2000), and in engineering signal processing (Cahoon and Hinich, 1976).

The critical issues of the sampling properties of parameter estimates for this statistical problem have been obscured by the dominance of this literature by the seminal work of measurement psychologists and psychometricians. A survey of the mathematics behind Schonemann's method is given by Sibson (1978). Schonemann's approach to metric unfolding based on squared distances does not formally treat the effects of random errors in the observations.

The new method presented in this paper deals with estimating the target locations and the observer positions when the observations are functions of the squared distances between observers and targets observed with an additive random error in a two dimensional space. The method is based on the work of Cahoon (1976) , Cahoon, Hinich and Ordeshook (1978) and Hinich (1978) . The Cahoon-Hinich (C-H) method is a statistical metric multidimensional unfolding method that is based on the multidimensional spatial theory of electoral competition originally developed by Davis and Hinich (1966). For a review of this theory and its extensions see Davis, Hinich and Ordeshook (1970) and Enelow and Hinich (1984).

The C-H method provides a method for using political survey data to make predictions about how candidates can adopt positions on critical issues in order to position themselves in the political space to maximize their vote in an upcoming election. The method has even been referenced in a study of strategic hospital planning by Drain and Godkin (1996).

The main result in this paper is a significant modification of the C-H method that yields much more robust estimates of the target locations given the parametric model presented next. The modification also yields more accurate estimates of the mean and variance of the observer locations $x_i$ than the original method. The statistical problem will be presented in terms of squared distances between a set of observers and targets. The method can switch between a straight distance model and squared distance model. For ease of exposition consider the squared distance model when the Euclidean space is two dimensional. The method is easily extended to Euclidean spaces whose dimension is larger than two but in applications to determine the nature of political spaces a variety of methods show that the spaces are almost always either *one or two dimensional.*

## 2 A STATISTICAL QUADRATIC DISTANCE MODEL

Suppose that there are $N$ observers and $M+1$ targets. Each observer at position $\mathbf{x}_i = (x_{i1}, x_{i2})'$ on a two dimensional surface reports the squared Euclidean distances $S(\boldsymbol{\pi}_m, \mathbf{x}_i)$ to the targets $m = 0, 1, \cdots, M$ at locations $\boldsymbol{\pi}_m = (\pi_{m1}, \pi_{m2})'$. Each reported distance has an additive stochastic error $e_{im}$ with mean $\beta v_m$ and variance $\psi_m^2$. The $v_m$ are assumed to be known but the scale parameter $\beta$ has to be estimated. Thus

(0.1) $$S(\boldsymbol{\pi}_m, \mathbf{x}_i) = (\boldsymbol{\pi}_m - \mathbf{x}_i)'(\boldsymbol{\pi}_m - \mathbf{x}_i) + e_{im} = \boldsymbol{\pi}_m'\boldsymbol{\pi}_m - 2\boldsymbol{\pi}_m'\mathbf{x}_i + \mathbf{x}_i'\mathbf{x}_i + e_{im}$$

Assuming that there are no missing distance reports there are $(M+1)N$ observations to estimate $2(M+1+N)$ observer and target positions. The parameter estimates will be derived from the $S(\boldsymbol{\pi}_m, \mathbf{x}_i)$ using a procedure that will be presented in the next section. There are enough observations and structure in the model to estimate all its parameters. The ability to incorporate the bias terms $\beta v_m$ in the unfolding problem is a unique feature of this method.

Before proceeding with this covariance based approach, consider the estimation of the target locations using the distances. If the errors are independently and normally distributed over observers and targets then the maximum likelihood parameter estimates $(\hat{\boldsymbol{\pi}}_m, \hat{\mathbf{x}}_i)$ would be the least-squares solution of the sum of the squares $\left(S(\hat{\boldsymbol{\pi}}_m, \hat{\mathbf{x}}_i) - S(\boldsymbol{\pi}_m, \mathbf{x}_i)\right)^2$ for $i = 1, \cdots, N$ and $m = 0, 1, \cdots, M$. This least-squares problem is nonlinear and the high dimensionality of the problem makes it computationally unfeasible.

The C-H method reduces the dimensionality of the problem by separating the estimation of the target positions from the estimation of the observer positions. The quadratic terms $\mathbf{x}_i'\mathbf{x}_i$ are removed by subtracting the distances to one target, say target $m = 0$ from the distances to the other targets and then computing the sample $MxM$ covariance matrix of the differences $D(\boldsymbol{\pi}_m, \mathbf{x}_i) = S(\boldsymbol{\pi}_m, \mathbf{x}_i) - S(\boldsymbol{\pi}_0, \mathbf{x}_i)$. The target whose distances are subtracted from the others is called the *reference target.* The importance of removing the quadratic terms will become clarified as the method is presented.

Since the origin of the space is not identified from the distance data and thus is arbitrary, the algebra is simplified by setting $\boldsymbol{\pi}_0 = 0$. Then

(0.2) $$D(\boldsymbol{\pi}_m, \mathbf{x}_i) = \boldsymbol{\pi}_m{}'\boldsymbol{\pi}_m - 2\boldsymbol{\pi}_m{}'\mathbf{x}_i + e_{im} - e_{i0}$$

The positions of the targets and the other parameters of the model are estimated from the sample covariance matrix of the $M$ differences $D(\boldsymbol{\pi}_m, \mathbf{x}_i)$'s.

Assume that the errors $e_m$ are independently and identically distributed and that they are independent of the observer positions $\mathbf{x}_i$. Assume that $x_{i1}$ and $x_{i2}$ are uncorrelated random variables whose variances are denote $\sigma_{x1}^2$ and $\sigma_{x2}^2$. Then the $MxM$ covariance matrix of the $D(\boldsymbol{\pi}_m, \mathbf{x}_i)$'s is

(0.3) $$\boldsymbol{\Sigma}_D = 4\boldsymbol{\Pi}\,\boldsymbol{\Sigma}_x\,\boldsymbol{\Pi}' + \boldsymbol{\Psi} + \psi_0^2\mathbf{1}$$

where $\boldsymbol{\Pi}' = (\boldsymbol{\pi}_1, \cdots, \boldsymbol{\pi}_M)$ is a $2xM$ matrix of target positions, $\boldsymbol{\Psi}$ is an $MxM$ diagonal matrix whose diagonal elements are the variances $\psi_m^2 = E(e_{im}^2)$ of the errors, $\psi_0^2$ is the variance of the error $e_{i0}$, $\mathbf{1}$ is an $MxM$ matrix of ones, and $\boldsymbol{\Sigma}_x = \begin{pmatrix} \sigma_{x1}^2 & 0 \\ 0 & \sigma_{x2}^2 \end{pmatrix}$ is the $2x2$ diagonal covariance matrix of the $\mathbf{x}_i = (x_{i1}, x_{i2})'$.

If the sample covariance matrix of the scores were used to estimate the model then there would be third and fourth order moments of the $x_i$ in the expected value of the scores covariances. For most application the observer locations will not have a symmetric distribution. A maximum likelihood factor analysis based on equation (2.3) is presented in the next section.

## 3 ESTIMATING THE TARGET LOCATIONS

To illustrate how maximum likelihood factor analysis can be applied to (2.3) assume for a while that $\psi_0$ is known. Then $\mathbf{\Sigma}_D - \psi_0^2 \mathbf{1}'\mathbf{1} = 4\mathbf{\Pi}\,\mathbf{\Sigma}_x\,\mathbf{\Pi}' + \mathbf{\Psi}$ is the $MxM$ covariance matrix of the $\boldsymbol{\pi}_m{}'\boldsymbol{\pi}_m - 2\boldsymbol{\pi}_m{}'\mathbf{x}_i + e_{im}$. This is a standard *factor analysis* model where the factor loading matrix is the $Mx2$ matrix $\mathbf{\Lambda} = 2\mathbf{\Pi}\mathbf{\Sigma}_x^{1/2}$ and $\mathbf{\Psi}$ is the $MxM$ diagonal matrix of additive error variances (Lawley and Maxwell, 1971).

The unbiased sample covariance matrix of the observation vectors $\mathbf{D}_i = \left(D(\boldsymbol{\pi}_1, \mathbf{x}_i), \cdots, D(\boldsymbol{\pi}_M, \mathbf{x}_i)\right)'$ is

(0.4) $$\mathbf{S} = \frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{D}_i - \bar{\mathbf{D}}\right)\left(\mathbf{D}_i - \bar{\mathbf{D}}\right)'$$

where $\bar{\mathbf{D}} = \left(\bar{D}(\boldsymbol{\pi}_1), \cdots, \bar{D}(\boldsymbol{\pi}_M)\right)'$ and $\bar{D}(\boldsymbol{\pi}_k) = \frac{1}{N}\sum_{i=1}^{N} D(\boldsymbol{\pi}_k, \mathbf{x}_i)$ is the sample mean of observations of the target k. Let $\hat{\mathbf{\Lambda}}(\psi_0)$ denote the maximum likelihood estimate of $\mathbf{\Lambda} = 2\mathbf{\Pi}\mathbf{\Sigma}_x^{1/2}$. If the observations are bounded then the asymptotic results of Anderson and Amemiya (1988) apply to the matrix $\mathbf{S} - \psi_0^2\mathbf{1}\mathbf{1}'$ and thus $\sqrt{N}\left[\hat{\mathbf{\Lambda}}(\psi_0) - \mathbf{\Lambda}(\psi_0)\right]$ is asymptotically normal as $N \to \infty$. Since the orientation of the two dimensional coordinate system is not identified from the model the estimated $Mx2$ factor loading matrix obtained from a maximum likelihood factor analysis of $\mathbf{S} - \psi_0^2\mathbf{1}\mathbf{1}'$ is $\hat{\mathbf{\Lambda}}(\psi_0) = 2\mathbf{\Pi}\mathbf{\Sigma}_x^{1/2}\mathbf{R} + \boldsymbol{\varepsilon}$ where the matrix $\mathbf{R} = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \sin\delta \end{pmatrix}$ is a $\delta$ angle orthogonal rotation of the coordinate system and $\boldsymbol{\varepsilon}$ is the error matrix of the estimate.

Joreskog (1967) shows that maximizing the likelihood is equivalent to minimizing the function $f(\boldsymbol{\Psi}) = \sum_{k=3}^{M} (\theta_k - \log\theta_k - 1)$ where $\theta_1 > \cdots > \theta_M$ are the ordered eigenvalues of the matrix $A(\boldsymbol{\Psi}) = \boldsymbol{\Psi}^{-1/2}\mathbf{S}\boldsymbol{\Psi}^{-1/2}$. This minimum is attained by finding the $\psi_{kk}$ that makes the $M-2$ smallest eigenvalues $\theta_k$ as close as possible to one using the least squares metric. Cahoon (1975) programmed the maximum likelihood algorithm of Clarke (1970) and I modified the algorithm to obtain the maximum likelihood estimates $\hat{\boldsymbol{\Lambda}}$ of $\boldsymbol{\Lambda} = 2\boldsymbol{\Pi}\boldsymbol{\Sigma}_x^{1/2}\mathbf{R}$, and the error variances $\psi_0^2, \psi_1^2, \ldots, \psi_M^2$.

This new approach to the target estimation problem is implemented in a FORTRAN 90 program that I call MAP. The complicated three dimensional rotations used in the C-H method is now eliminated as is the need for the first least squares fit.

The rotation angle $\delta$, the elements of the mean ideal point $\boldsymbol{\mu}_x = E(\mathbf{x}_i)$, and the variances $\sigma_{x1}^2$ and $\sigma_{x2}^2$ are not identified from the structure in expression (2.3). These parameters are not estimable using any method that is only a function of the sample covariance. They are estimable using the vector of the sample means $\bar{\mathbf{D}} = (\bar{D}(\boldsymbol{\pi}_1), \cdots, \bar{D}(\boldsymbol{\pi}_M))'$, as is shown next. Once these parameters are estimated then the estimate of the target location matrix is $\hat{\boldsymbol{\Pi}} = \frac{1}{2}\hat{\mathbf{R}}'\hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{\Sigma}}_x^{-1/2}$ where $\hat{\boldsymbol{\Sigma}}_x = \begin{pmatrix} \hat{\sigma}_{x1}^2 & 0 \\ 0 & \hat{\sigma}_{x2}^2 \end{pmatrix}$ is the estimated covariance matrix and $\hat{\mathbf{R}}$ is the estimated rotation.

## 4 ESTIMATING THE REMAINING PARAMETERS

If the covariance matrix $\boldsymbol{\Sigma}_x$ of the observer locations $\mathbf{x}_i$ is diagonal with diagonal elements $\sigma_{x1}^2 \neq \sigma_{x2}^2$ then the north-south orientation of the axes are identified up to $180^o$ rotations since the covariance matrix $\boldsymbol{\Sigma}_x$ is no longer diagonal if the axes are rotated. To formalize this assertion note that from expression (2.2) it follows that the expected value of each difference is $E[D(\boldsymbol{\pi}_m, \mathbf{x}_i)] = \boldsymbol{\pi}_m{}'\boldsymbol{\pi}_m - 2\boldsymbol{\pi}_m{}'\boldsymbol{\mu}_x + \beta v_m$ where $\boldsymbol{\mu}_x = E(\mathbf{x}_i)$ and $v_m$ is the bias of the stochastic error.

Let $\mathbf{u}_m = (0,0,\cdots,1,0\cdots0)'$ where the one is at the $m^{th}$ position in the $M$ dimensional vector. Since $\boldsymbol{\Pi} = \frac{1}{2}\boldsymbol{\Sigma}_x^{-1/2}\mathbf{R}'$ because $\mathbf{R}^{-1} = \mathbf{R}'$ then $4\boldsymbol{\pi}_m{}'\boldsymbol{\pi}_m = \mathbf{u}_m{}'\boldsymbol{\Lambda}\mathbf{R}'\boldsymbol{\Sigma}_x^{-1/2}\mathbf{R}\boldsymbol{\Lambda}\mathbf{u}_m$. Thus

(0.5)
$$\boldsymbol{\pi}_m{}'\boldsymbol{\pi}_m = \frac{1}{4}\sum_{i=1}^{2}\sigma_{xi}^{-1}\left(\sum_{k=1}^{2}\lambda_{mk}r_{ik}\right)^2$$

where $\lambda_{mk}$ and $r_{mk}$ are the $mk^{th}$ elements of the matrices $\boldsymbol{\Lambda}$ and $\mathbf{R}$. Similarly

(0.6)
$$\boldsymbol{\pi}_m{}'\boldsymbol{\mu}_x = \frac{1}{2}\sum_{i=1}^{2}\sigma_{xi}^{-1}\mu_{xi}\left(\sum_{k=1}^{2}\lambda_{mk}r_{ik}\right)^2$$

and thus it follows that

(0.7)
$$E\left[D(\boldsymbol{\pi}_m, \mathbf{x}_i)\right] = -\alpha_1\lambda_{m1}^2 - \alpha_2\lambda_{m2}^2 - \alpha_3\lambda_{m1}\lambda_{m2} + \alpha_4\lambda_{m1} + \alpha_5\lambda_{m2} + \beta v_m$$

where

(0.8)
$$\alpha_1 = \frac{\cos^2\delta}{\sigma_{x1}^2} + \frac{\sin^2\delta}{\sigma_{x2}^2} \qquad \alpha_2 = \frac{\sin^2\delta}{\sigma_{x1}^2} + \frac{\cos^2\delta}{\sigma_{x2}^2} \qquad \alpha_3 = \left(\frac{1}{\sigma_{x2}^2} - \frac{1}{\sigma_{x1}^2}\right)\sin(2\delta)$$
$$\alpha_4 = 2\left(\frac{\mu_{x1}\cos\delta}{\sigma_{x1}} + \frac{\mu_{x2}\sin\delta}{\sigma_{x2}}\right) \qquad \alpha_5 = 2\left(-\frac{\mu_{x1}\sin\delta}{\sigma_{x1}} + \frac{\mu_{x2}\cos\delta}{\sigma_{x2}}\right)$$

The rotation angle $\delta$, the population mean ideal point $\boldsymbol{\mu}_x$, the variances $\sigma_{x1}^2$ and $\sigma_{x2}^2$ and the bias scale parameter $\beta$ are estimated from an errors-in variables least squares fit of the $\bar{D}(\boldsymbol{\pi}_m)$ to the estimates $\hat{\lambda}_{mk}$ of the $Mx2$ elements $\lambda_{mk}$ of the matrix $\boldsymbol{\Lambda}$ and the $v_m$ using the plug-in model

$$\bar{D}(\boldsymbol{\pi}_m, \mathbf{x}_i) = -\alpha_1 \hat{\lambda}_{m1}^2 - \alpha_2 \hat{\lambda}_{m2}^2 - \alpha_3 \hat{\lambda}_{m1} \hat{\lambda}_{m2} + \alpha_4 \hat{\lambda}_{m1} + \alpha_5 \hat{\lambda}_{m2} + \beta v_m \tag{0.9}$$

The parameter estimates are

$$\begin{aligned} &\hat{\delta} = \frac{1}{2}\tan^{-1}\left(\frac{\hat{\alpha}_3}{\hat{\alpha}_2 - \hat{\alpha}_1}\right) \quad \hat{\sigma}_{x1}^2 = \frac{1}{2}\left(\hat{\alpha}_1 + \hat{\alpha}_2 - \frac{\hat{\alpha}_3}{\sin\left(2\hat{\delta}\right)}\right) \quad \hat{\sigma}_{x2}^2 = \frac{1}{2}\left(\hat{\alpha}_1 + \hat{\alpha}_2 + \frac{\hat{\alpha}_3}{\sin\left(2\hat{\delta}\right)}\right) \\ &\hat{\mu}_{x1} = \frac{\hat{\sigma}_1}{2}\left(\hat{\alpha}_4 \cos\hat{\delta} - \hat{\alpha}_5 \sin\hat{\delta}\right) \qquad \hat{\mu}_{x2} = \frac{\hat{\sigma}_2}{2}\left(\hat{\alpha}_4 \sin\hat{\delta} + \hat{\alpha}_5 \cos\hat{\delta}\right) \end{aligned} \tag{0.10}$$

Note that that these estimates are nonlinear functions of the least squares estimates $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4, \hat{\alpha}_5)$ obtained from the errors-in-variables regression (4.5). These estimates are biased and the biases propagate through the nonlinear transformations.

The $\hat{\lambda}_{mk}$ are maximum likelihood estimates of the $\lambda_{mk}$ but the estimates in expression (4.6) are subtly biased for both finite $N$ and asymptotically due to the errors-in-variables. The errors go to zero as $N$ goes to infinity but the covariance matrix of the estimates is not diagonal and thus the errors in the right hand side of (4.5) propagate in a complicated manner to the $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4, \hat{\alpha}_5)$ which then propagates to the estimates of the remaining parameters.

The parameter estimates are bootstrapped by resampling the data with replacement one hundred times. The mean and standard error of each parameter estimate are computed from the one hundred resampled estimates.

The simulation results presented next show that there is a bias in the target map even for large samples due to the errors-in-variables in the least square fit of (4.5). This method takes as much advantage of the statistical squared distance model as can be achieved. The means and the standard errors of this method will be demonstrated next using simulations.

## *5 SAMPLING STATISTICS FROM SIMULATIONS*

I wrote a FORTRAN 90 program to produce artificial data to feed the MAP program that implements the estimates that I just described. The results that are presented next are for a configuration of 25 targets that are displayed in Figure 1. The additive errors are a set of scaled pseudorandom independent normal variates and the observer positions are another set of pseudorandom independent normal variates. These pseudorandom normals were generated by a call to a subroutine that I wrote that uses a good congruential generator to return a vector of pseudorandom independent uniform (0, 1) variates. The standard deviations of the observer positions are $\sigma_{x1} = 5$ and $\sigma_{x2} = 2$. The error variances were set equal.

To keep the experiment manageable only two values of the sample size are used: $N = 200$ and $N = 200,000$. The two values of the error standard deviation $\psi = \psi_0 = \cdots = \psi_M$ are $\psi = 0.1$ and $\psi = 1$. The measure of goodness of fit is the root-mean square error (RMSE) of the estimated target locations with respect to the true positions in Figure 1. The best results are obtained by using the target at the origin as the reference target.

Consider the case when all the scale parameter $\beta$ is zero. For $N = 200$ the RMSE is 2.85 for $\psi = 0.1$ and 2.74 for $\psi = 1$. The maximum absolute errors are 14.27 and 13.72 respectively. Since the bootstrapped standard error of the rotation is 1.68 and 1.23 for $\psi = 0.1$ and $\psi = 1$ respectively, the differences between the RMSE values are not statistically significant at the 5% level. The error in the rotation is about 4.1 for both error variances and this error is the main source for the error of the estimated target map.

For $N = 200,000$ the RMSE is 2.70 for both $\psi = 0.1$ and for $\psi = 1$. The maximum absolute errors are 13.48 for both. The larger sample size did not yield a significantly better fit as compared with $N = 200$.

For $N = 200$ the errors of the estimates of the coordinates of the mean observer positions are $\hat{\mu}_{x1} - \mu_{x1} = -0.26$ and $\hat{\mu}_{x2} - \mu_{x2} = 0.25$ for $\psi = 0.1$, and $\hat{\mu}_{x1} - \mu_{x1} = -0.25$ and $\hat{\mu}_{x2} - \mu_{x2} = 0.24$ for $\psi = 1$. The bootstrapped means are very similar to the true estimates. The bootstrapped standard errors are 0.07 for $\hat{\mu}_{x1}$ and 0.06 for $\hat{\mu}_{x2}$ for both error variances. Thus the errors of $\hat{\mu}_{x1}$ and $\hat{\mu}_{x2}$ are not statistically significant for both error variances at the 0.1% level.

For $N = 200,000$ the errors are $\hat{\mu}_{x1} - \mu_{x1} = -0.08$ and $\hat{\mu}_{x2} - \mu_{x2} = 0.05$ for both error variances. The bootstrapped standard errors are 0.008 for $\hat{\mu}_{x1}$ and 0.003 for $\hat{\mu}_{x2}$ and thus the errors are statistically significant at the 0.1% level. The larger sample size improved the accuracy of the estimate of the mean ideal point but the bias is more statistically significant.

For $N = 200$ the errors of the ideal point standard deviations are $\hat{\sigma}_{x1} - \sigma_{x1} = 0.05$ and $\hat{\sigma}_{x2} - \sigma_{x2} = -0.15$ for both error variances. The bootstrapped standard errors are 0.27 and 0.09 for both error variances. Thus the errors are not statistically significant at the 5% level. The errors are $\hat{\sigma}_{x1} - \sigma_{x1} = 0.01$ and $\hat{\sigma}_{x2} - \sigma_{x2} = 0.00$ for both error variances and $N = 200,000$. The bootstrapped standard errors are 0.008 and 0.003 for both error variances. Once again the errors are not statistically significant.

Now consider the results when $\beta = 1$ and the $v_m$ values of the errors have a pseudorandom normal distribution with a zero mean and unit variance. For $N = 200$ the RMSE is 2.91 for $\psi = 0.1$ and 2.85 for $\psi = 1$. The maximum absolute errors are 14.56 and 14.24 respectively. These results are statistically the same as when $\beta = 0$. The same holds for the other parameter estimates for all the cases, which is not surprising since the addition of one independent variable that is almost uncorrelated with the other five variables will not significantly change the other five estimates.

The error for the estimate of the scale parameter for $N = 200$ is $\hat{\beta} - \beta = 0.18$ for $\psi = 0.1$ and $\hat{\beta} - \beta = 0.06$ for $\psi = 1$. The bootstrapped standard errors are 0.01 and 0.12 respectively. The error in $\hat{\beta}$ for $\psi = 1$ is not statistically significant at the 5% level but it is statistically significant at the 0.1% for $\psi = 0.1$.

The errors become much larger if the reference target is not near the origin. If the reference target is the point $T_a = (11.55, \text{-}4.01)$ in Figure 1 and $\beta = 0$. Then the RMSE is 34.8 for $N = 200$ and $\psi = 0.1$ rather than 2.85 for the origin reference target. The maximum absolute error is 174.0. The errors of the estimates of the coordinates of the mean observer positions are $\hat{\mu}_{x1} - \mu_{x1} = 4.5$ and $\hat{\mu}_{x2} - \mu_{x2} = 5.72$ for $\psi = 0.1$, which are more than ten times the errors when the origin is the reference target. The reason for the larger differences between the true positions of the targets and their estimates is the increased error in the least squares estimates of $\alpha_4$ and $\alpha_5$. The estimates for the zero reference are $\hat{\alpha}_4 = 0.88$ and $\hat{\alpha}_5 = 0.71$ whereas the estimates for the $T_a$ reference are $\hat{\alpha}_4 = 4.57$ and $\hat{\alpha}_5 = 5.65$. The first three estimates are the same for both reference targets.

When the reference target is $T_b = (\text{-}0.19, 8.77)$ the errors of the estimates of the coordinates of the mean observer positions are $\hat{\mu}_{x1} - \mu_{x1} = 1.79$ and $\hat{\mu}_{x2} - \mu_{x2} = 8.85$ for $\psi = 0.1$. The RMSE is 27.77 and the maximum absolute error is 138.86. The estimates for fourth and fifth least squares estimates are $\hat{\alpha}_4$=-0.33 and $\hat{\alpha}_5 = 11.2$. The fourth estimate has the wrong sign and the fifth is much larger than the estimate when the origin is the reference. There is no way to eliminate the errors since these parameters are identified using the equations in (4.5) and both the independent and dependent variables have stochastic error components.

## 6 ESTIMATING THE OBSERVER LOCATIONS

The $i^{th}$ observer location is estimated by a least squares fit of the linear system

$$\bar{D}(\boldsymbol{\pi}_m) = \hat{\boldsymbol{\pi}}_m'\hat{\boldsymbol{\pi}}_m - 2\hat{\boldsymbol{\pi}}_m'\mathbf{x}_i \tag{0.11}$$

which is a sample version of expression (2.2). This fit is also an errors-in-variables least squares and so the estimates are biased. To test the accuracy of the estimation the simulation program calculated the percent of the true $\mathbf{x}_i$'s and the estimated $\hat{\mathbf{x}}_i$'s that are closest to each target. For $N = 200$ and $\psi = 0.1$ the largest true percent was 28% for target 22 and the percentage for the estimates was 20% yielding an 8% error. The second largest true percentage was 16.5% for target 21 and the percentage for the estimates was 9% yielding a 7.5% error. The target at the origin, the reference target, had a true percentage of 15% and so did the estimates. The errors for the other percentages were smaller than 7.5%. The results for $\psi = 1$ were surprisingly slightly better than for the smaller error variance.

The differences for $N = 200{,}000$ were smaller. For $\psi = 1$ the percentage for the target with the largest true percentage was 22.5% yielding a 5.5% error. The percentage for the reference target was 19.7% yielding an error of 4.2%. For $\psi = 0.1$ the maximum error was 3.5% and the rest were about 1%.

## *7 A TURKISH POLITICS EXAMPLE*

The Cahoon-Hinich (C-H) method is used to estimate the positions of the voters and the candidates or parties in a latent political space depending on the political system of the democracy in question using a set of survey questions.

The application of the method to the spatial theory of politics requires a set of assumptions relating the data to a spatial model. First, the scores given to each party is assumed to be a monotonically decreasing function of the Euclidian distance between the position of the party in the space and the most preferred ideological position of the respondent. This position is called the ideal point. The respondent is not required to articulate that position but rather it is a latent position in the latent space. Second, the constellation of the party positions in the latent space is assumed to be the same across all respondents. The only thing that differs from respondent to respondent is their personal ideal points. The method is then applied to these scores that we get from the respondents.

The method was applied to a data set from a public opinion survey taken in 2001. For a complete description of the survey and the analysis see Çarkoğlu and Hinich (2003). Our data comes from a nation-wide representative survey of urban population conducted during the chaotic weeks of the second economic crisis of February 2001. A total of 1201 face-to-face interviews were conducted in 12 of the 81 provinces. The questionnaires were administered between February 20 and March 16 2001 using a "random sampling" method with an objective to represent the nation-wide voting age urban population living within municipality borders, in which the urban population figures of 1997 census data were taken as the basis.

Each respondent was asked to grade the seven major parties in terms of how well that party would impact on the respondent's family is the party were to receive a majority of the seats in the parliament. These parties obtained 94.8 percent of the urban vote in 1999 elections. However as of February-March 2001 these parties comprise only the preferences of 42.3 percent of our urban sample. Similar to opinion poll results reported in the media, our findings also indicate that while 6 percent of the respondents report that they will not cast their vote and about 5 percent are undecided as to which party to vote for. More significantly nearly 33 percent of the respondents indicate that they will not cast their vote for any one of the existing parties. Given the continual crisis atmosphere in the country, the erosion of electoral support for the coalition partners, which amounts to a total of about 39 percentage points in the urban areas, is not surprising. Among the opposition only the left leaning CHP party and pro-Kurdish HADEP party maintained their urban constituencies. The rest of the opposition parties have lost their supporters.

The results of a two-dimensional latent ideological spatial map of these parties together with the estimated respondents' ideal points are presented in the Figure 2. The horizontal axis appears to posit the pro-Islamist FP in one extreme as opposed to the secularist left leaning CHP. The relative positions of the rest of the parties fit our expectations about the religious cleavage in Turkish politics. The nationalist MHP turns out to be the closest one to the position of the pro-Islamist FP on this axis. Among the centrist parties DYP is slightly closer to the pro-Islamist end. DSP and CHP are clustered together on the opposing end of this dimension placed to the left of ANAP's centrist position. It is noticeable that HADEP's position on this dimension is closer in the perceptions of our respondents to the secularist left of DSP and CHP.

The vertical axis has the Kurdish HADEP on one extreme and the nationalist MHP and DSP on the other. While ANAP, CHP and FP's positions come close to the center on this dimension, DYP is placed closer to the nationalist MHP and DSP's opposing end. It has been suggested that FP's strong showing in the East and Southeastern provinces where the bulk of Kurdish population lives is evidence of FP's appeal to the Kurdish electorate. Similarly the religiously conservative Kurdish constituency was seen by many Turkish politics scholars as a cause for ideological closeness of HADEP and FP. Our spatial map clearly shows that in the perceptions of the urban population, HADEP is nowhere close to the position of FP on the two-dimensional political space we induce from the data using MAP.

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**Figure 1   25 Test Targets**

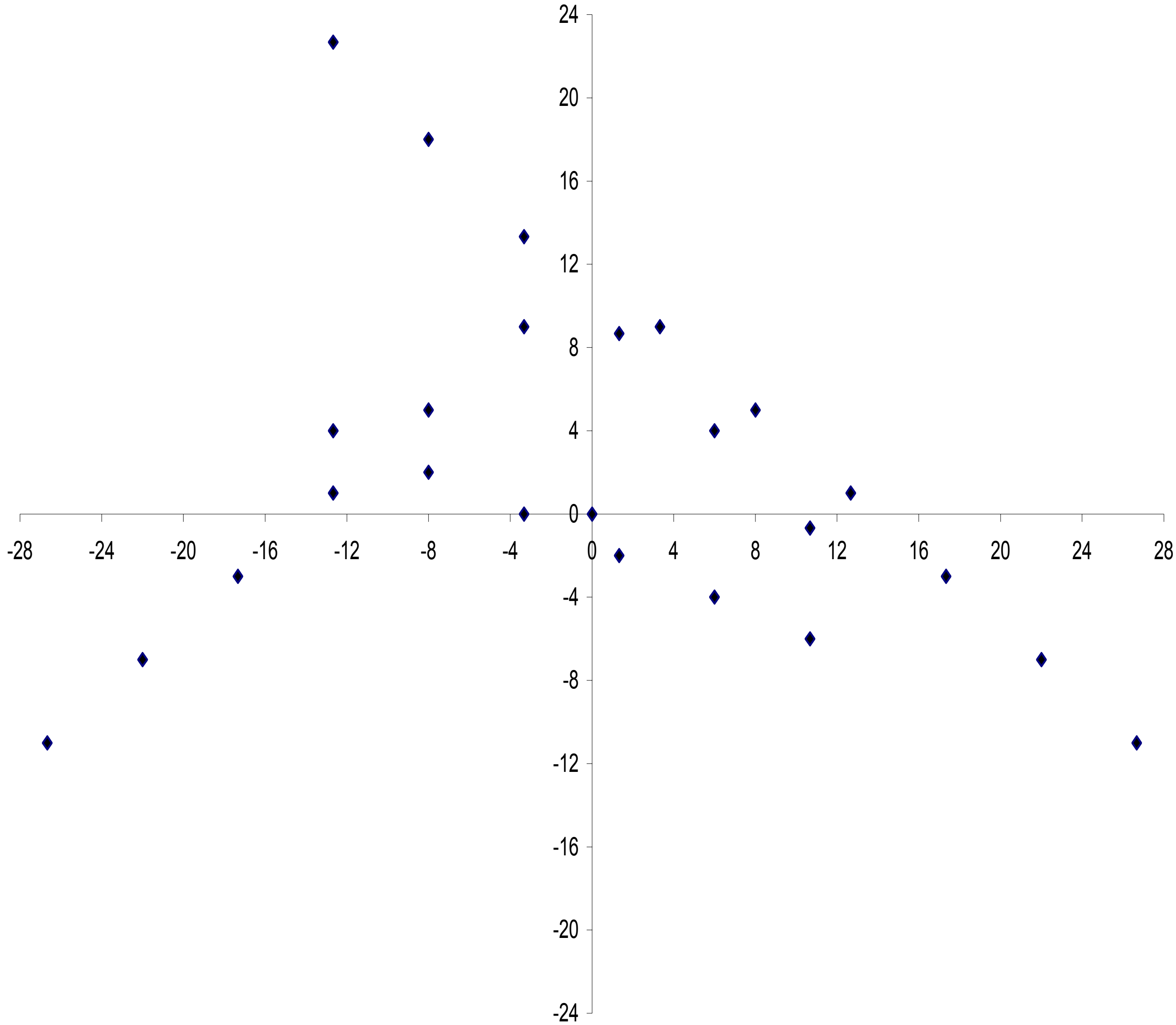

**Figure 1. Estimated ideal points and party positions, full sample**