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FREQUENCY-DOMAIN TEST OF TIME REVERSIBILITY

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We introduce a frequency-domain test of time reversibility, the REVERSE test. It is based on the bispectrum. We analytically establish the asymptotic distribution of the test and also explore its finite-sample properties through Monte-Carlo simulation. Following other researchers who demonstrated that the problem of business-cycle asymmetry can be stated as whether macroeconomic fluctuations are time irreversible, we use the REVERSE test as a frequency-domain test of business-cycle asymmetry. Our empirical results show that time irreversibility is the rule rather than the exception for a representative set of macroeconomic time series for five OECD countries.

Keywords: Nonlinearity, Asymmetry, Business Cycle, Bispectrum

1. INTRODUCTION

Economists have long been interested in the problem of business-cycle asymmetry, especially since the pioneering work by Burns and Mitchell (1946). Through informal statistical analysis, these authors documented that many U.S. economic time series appear to exhibit periods of expansion that are longer and slower than their subsequent contractionary phases. The modern empirical literature on this problem was opened by Neftci (1984), who studied the time-symmetry properties of the U.S. aggregate unemployment rate. In the many papers that followed, various asymmetry metrics were introduced and analyzed.

Recently, Ramsey and Rothman (1996) showed that the question of business-cycle asymmetry can be restated as whether the dynamic behavior of key macroeconomic variables is time reversible. If the probabilistic structure of a time series going forward is identical to that in reverse, the series is said to be time reversible. If the series is not time reversible, it is said to be time irreversible. Having established the connection between time reversibility and business-cycle asymmetry, Ramsey and Rothman (1996) developed and applied a time-domain test of time reversibility called the Time Reversibility (TR) test.

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We introduce a frequency-domain test of time reversibility, based on the bispectrum and called the REVERSE test. This test is new to the statistical time-series literature and exploits a property of higher-order spectra for time-reversible processes, i.e., the imaginary part of all polyspectra is zero for time-reversible stochastic processes. The REVERSE test is related to Hinich's (1982) Gaussianity test. In particular, it checks whether the breakdown of Gaussianity is due to time irreversibility.

The REVERSE test complements the time-domain TR test of Ramsey and Rothman (1996). Both the REVERSE and TR tests examine the behavior of estimated third-order moments to check for departures from time reversibility. The REVERSE test, however, explores more of the third-order moment structure and, as a result, may have higher power than the TR test against important time-irreversible alternatives. Further, the analysis of the variance of the REVERSE test statistic is arguably more complete than the analogous analysis for the TR test.

The hypothesis that we wish to falsify is that the time series is a realization from a stationary random time-reversible process with finite moments. This hypothesis implies that the bicovariance function is symmetric, which in turn implies that the imaginary part of the bispectrum is zero. There is another null hypothesis that implies a similar result, namely, that the process is linear with innovations drawn from a symmetric distribution. Such a linear process will have zero bicovariances, and thus zero bispectrum.

The issue of time reversibility is most important for nonlinear processes, because most interesting nonlinear stochastic processes are not time reversible. Thus, the detection of time reversibility is part of the general research program of detecting and identifying a nonlinear process using time-series analysis. For example, evolutionary processes are path dependent and do not return to the initial position with certainty.

Following the argument of Ramsey and Rothman (1996), as a test of time reversibility the REVERSE test also serves as a frequency-domain test of business-cycle asymmetry. We apply the REVERSE test to monthly data, from the early 1960's to the mid-1990's, for representative macroeconomic time series for five Organization for Economic Cooperation and Development (OECD) countries. Our results provide ubiquitous evidence that business-cycle fluctuations over the past 35 years are time irreversible, establishing asymmetry of business-cycle movements as a stylized fact for several of the world's largest economies.

The paper proceeds as follows. Section 2 provides a formal definition of time reversibility and introduces the REVERSE test. We establish the asymptotic distribution of the REVERSE test statistic in that section. Small sample properties of the test are analyzed in Section 3. In Section 4, we discuss the relationship between business-cycle asymmetry and time reversibility. We apply the REVERSE test to our international data set in Section 5. We conclude in Section 6.

2. TIME REVERSIBILITY AND THE REVERSE TEST

2.1. Time-Reversible Time Series

A formal statistical definition of time reversibility follows.

DEFINITION. *A time series* $\{x(t)\}$ *is time reversible if for every positive integer* n*, every* $t_1, t_2, \ldots, t_n \in \mathbf{R}$*, and all* $n \in \mathbf{N}$*, the vectors* $[x(t_1), x(t_2), \ldots, x(t_n)]$ *and* $[x(-t_1), x(-t_2), \ldots, x(-t_n)]$ *have the same joint probability distributions.*

Under this definition, one can show that time reversibility implies stationarity. Likewise, nonstationarity implies time irreversibility. In what follows, we abstract from problems of nonstationarity-induced time irreversibility and restrict ourselves solely to stationary time-irreversible stochastic processes.

Clearly, $\{x(t)\}$ is time reversible when $\{x(t)\}$ is independently and identically distributed (i.i.d.). The result that stationary Gaussian processes are time reversible appeared as Theorem 1 in Weiss (1975, p. 831). In the same paper, Weiss proved the converse within the context of discrete-time ARMA models. Hallin et al. (1988) extended this result to the case of general linear processes.

It is straightforward to show that time irreversibility can stem from two sources: (1) The underlying model may be nonlinear even though the innovations are symmetrically (perhaps normally) distributed; or (2) the underlying innovations may be drawn from a non-Gaussian probability distribution while the model is linear. Nonlinearity does not imply time irreversibility; there exist stationary nonlinear time processes that are time reversible. A test for time irreversibility, then, is not equivalent to a test for nonlinearity.[1]

2.2. REVERSE Test

Let $\{x(t)\}$ be a real-valued mean-zero third-order stationary time series. The general third-order moments $c_x(s, r)$ are defined as follows:

$$c_x(r, s) = E[x(t+r)x(t+s)x(t)], \qquad s \le r, \qquad r = 0, 1, 2, \ldots. \tag{1}$$

The bispectrum is the double Fourier transform of the third-order cumulant function. More specifically, the bispectrum is defined for frequencies f_1 and f_2 in the domain

$$\Omega = \{(f_1, f_2) : 0 < f_1 < 0.5, \qquad f_2 < f_1, \qquad 2f_1 + f_2 < 1\}, \quad \text{as}$$

$$B_x(f_1, f_2) = \sum_{t_1=-\infty}^{\infty} \sum_{t_2=-\infty}^{\infty} c_x(r, s) \exp[-i2\pi(f_1 r + f_2 s)]. \tag{2}$$

If $\{x(t)\}$ is time reversible, then

$$c_x(r, s) = c_x(-r, -s), \tag{3}$$

so that the imaginary part of the bispectrum $B_x(f_1, f_2)$ is zero if $\{x(t)\}$ is time reversible. The result that the imaginary part of the bispectrum is zero for time-reversible processes follows from Brillinger and Rosenblatt's (1967) discussion of the relationship between time reversibility and polyspectra.

The skewness function $\Gamma(f_1, f_2)$ is defined in terms of the bispectrum as follows:

$$\Gamma^2(f_1, f_2) = |B_x(f_1, f_2)|^2 \big/ [S_x(f_1)S_x(f_2)S_x(f_1 + f_2)], \tag{4}$$

where $S_x(f)$ is the spectrum of $\{x(t)\}$ at frequency f. If $\{x(t)\}$ is time reversible, then the imaginary part of $\Gamma(f_1, f_2)$ is zero at all bifrequencies.

Time reversibility can be tested using a sample estimator of the skewness function $\Gamma(f_1, f_2)$. We next outline the procedure that we use to estimate the bispectrum.

Divide the sample $\{x(0), x(1), \ldots, x(N-1)\}$ into non-overlapping frames of length L and define the discrete Fourier frequencies as $f_k = k/L$. If N is not divisible by L, where N is the sample size, the last incomplete frame's data are not used. Thus, the number of frames used, P, is given by $P = (N/L]$, where the brackets denote integer division. The resolution bandwidth, δ, is defined as $\delta = 1/L$. For the pth frame of length L, $p = 0, 1, \ldots, P-1$, calculate

$$Y\left(f_{k_1}, f_{k_2}\right) = X\left(f_{k_1}\right) X\left(f_{k_2}\right) X^*\left(f_{k_1} + f_{k_2}\right), \tag{5}$$

where

$$X(f_k) = \sum_{t=0}^{L-1} x[t + (p \cdot L)] \exp\{-i 2\pi f_k [t + (p \cdot L)]\}. \tag{6}$$

Let $\langle B_x(f_{k_1}, f_{k_2})\rangle$ denote a smoothed estimator of $B_x(f_1, f_2)$; $\langle B_x(f_{k_1}, f_{k_2})\rangle$ is obtained by averaging over the values of $Y(f_{k_1}, f_{k_2})/L$ across the P frames. Theorem A.3 of the Appendix shows that this is a consistent and asymptotically complex normal estimator of the bispectrum, $B_x(f_1, f_2)$, if the sequence (f_{k_1}, f_{k_2}) converges to (f_1, f_2). The large sample variance of $\langle B_x(f_{k_1}, f_{k_2})\rangle$ is

$$\mathrm{Var} = [1/(\delta^2 N)] \left\langle S_x\left(f_{k_1}\right)\right\rangle \left\langle S_x\left(f_{k_2}\right)\right\rangle \left\langle S_x\left(f_{k_1} + f_{k_2}\right)\right\rangle, \tag{7}$$

where $\langle S_x(f)\rangle$ is defined as a consistent and asymptotically normal estimator of the power spectrum at frequency f and δ is the resolution bandwidth set in calculating $\langle B_x(f_{k_1}, f_{k_2})\rangle$.

The normalized estimated bispectrum is

$$A\left(f_{k_1}, f_{k_2}\right) = \sqrt{P/L} \left\langle B_x\left(f_{k_1}, f_{k_2}\right)\right\rangle \big/ \mathrm{Var}^{1/2}. \tag{8}$$

Let $\operatorname{Im} A(f_{k_1}, f_{k_2})$ denote the imaginary part of $A(f_{k_1}, f_{k_2})$. The REVERSE test statistic is defined as

$$\mathrm{REVERSE} = \sum_{(k_1,k_2)\in D}\sum \left|\operatorname{Im} A\left(f_{k_1}, f_{k_2}\right)\right|^2, \quad \text{where}$$

$$D = \left\{(k_1, k_2) : \left(f_{k_1}, f_{k_2}\right) \in \Omega\right\}. \tag{9}$$

Under the null hypothesis of time reversibility, so that $\operatorname{Im} B_x(f_1, f_2) = 0$ for all bifrequencies, Theorem A.4 of the Appendix shows that the REVERSE test statistic is distributed central chi squared with $M = [L^2/16]$ degrees of freedom.

Table 1. Estimated sizes of the REVERSE test statistic at 5% and 1% nominal size[a]

Simulated series	Estimated size at 5% nominal size	Estimated size at 1% nominal size
A. Independently and identically distributed processes		
Standard normal distribution	0.051	0.009
Uniform distribution on the unit interval	0.011	0.001
B. Gaussian AR models fitted to industrial production index growth rates for:		
Canada	0.051	0.011
Germany	0.053	0.013
Japan	0.066	0.015
United Kingdom	0.049	0.010
United States	0.067	0.017
C. Gaussian AR models fitted to inflation rates for:		
Canada	0.056	0.012
Germany	0.051	0.010
Japan	0.049	0.010
United Kingdom	0.053	0.011
United States	0.062	0.015
D. Gaussian AR models fitted to first differences of unemployment rates for:		
Canada	0.055	0.012
Germany	0.073	0.019
Japan	0.053	0.012
United Kingdom	0.076	0.019
United States	0.054	0.012

[a] Results are based on Monte-Carlo simulations with 10,000 iterations. In each iteration, a realization of sample size 414 of the stochastic process was generated and the REVERSE test statistic was calculated; 414 is the representative sample size of the OECD time series analyzed in Tables 2, 3, and 4. The estimated sizes were calculated as the fraction of rejections across the 10,000 iterations at both the 5% and 1% nominal size levels.

The same theorem also establishes the consistency of the test if Im $B_x(k_1, k_2) \neq 0$, if $P/L \to \infty$ as $P \to \infty$.

3. FINITE-SAMPLE PROPERTIES

Table 1 reports Monte-Carlo results on the small-sample properties of the REVERSE test statistic. These results were obtained through simulations run with 10,000 iterations. In each iteration, a series of sample size 414 of the particular stochastic process was generated and the REVERSE test statistic was calculated; 414 is the representative sample size of the OECD macroeconomic time series analyzed in Section 5. The estimated sizes in each simulation were calculated as the fraction of rejections across the 10,000 iterations at both the 5% and 1% nominal size levels.

Table 1A reports estimated sizes for two i.i.d. cases: (1) the standard normal distribution; and (2) the uniform distribution on the unit interval. For the standard normal distribution, the REVERSE test statistic has converged to its asymptotic distribution at the sample size considered. In the uniform case, however, the REVERSE test is conservative at this sample size; e.g., with 5% nominal size, the test rejects only 1.1% of the time.

To further explore the small-sample behavior of the REVERSE test, we estimated the size of the test for a set of finite-order autoregressive (AR) models. Given our interest in testing for evidence of business-cycle asymmetry with monthly OECD data, we designed the experiment as follows. For each OECD time series, we identified an AR model with the Akaike information criterion (AIC). Using Gaussian innovations, so that the stochastic process is time reversible, for each AIC-identified AR model, we ran a simulation to estimate the size of the REVERSE test. Our results appear in panels B, C, and D of Table 1, which report the estimated sizes for AR models fitted to the monthly OECD industrial production growth rates, inflation rates, and first differenced unemployment rates, respectively.

In most cases the estimated sizes match the nominal sizes very closely for these Gaussian AR models. The strongest size distortions occur for the AR models fitted to the first-differenced unemployment rates for Germany and the United Kingdom. For these two AR processes, however, the size distortion is nonetheless relatively modest; e.g., at the 5% nominal size level, the REVERSE test rejects slightly greater than 7% of the time. Our Monte-Carlo simulations thus establish that the REVERSE test is well behaved at the sample size considered in our application. Thus, chances are remote that rejections obtained with our test are spurious.

4. BUSINESS-CYCLE ASYMMETRY AND TIME IRREVERSIBILITY

The fundamental question addressed in the business-cycle asymmetry literature is whether macroeconomic fluctuations shift across business-cycle phases in a manner that is inconsistent with conventional theoretical and empirical models. DeLong and Summers (1986, p. 167) note that “statistical models of the sort used in economics . . . are entirely unable to capture cyclical asymmetries. If, as Keynes, Mitchell, and others believed, cyclical asymmetries are of fundamental importance, then standard statistical techniques are seriously deficient.” Standard time-series tools used by macroeconomists assume, for example, that impulse response functions are invariant with respect to the stage of the business cycle. However, the reaction of macroeconomic time series to shocks during an expansionary phase may be significantly different than during a contractionary phase.[2]

Building on Sichel’s (1993) analysis, Ramsey and Rothman (1996) introduced the notion of longitudinal asymmetry. Longitudinal asymmetry captures the idea of temporal asymmetric behavior in the direction of the business cycle. Such asymmetry is depicted in the plot of the slow-up and fast-down time series found in Figure 1 and captures what Sichel (1993) identified as steepness asymmetry.

Various approaches have been followed in the literature to detect such asymmetric behavior. In each, the major theme has been to identify a feature of the data

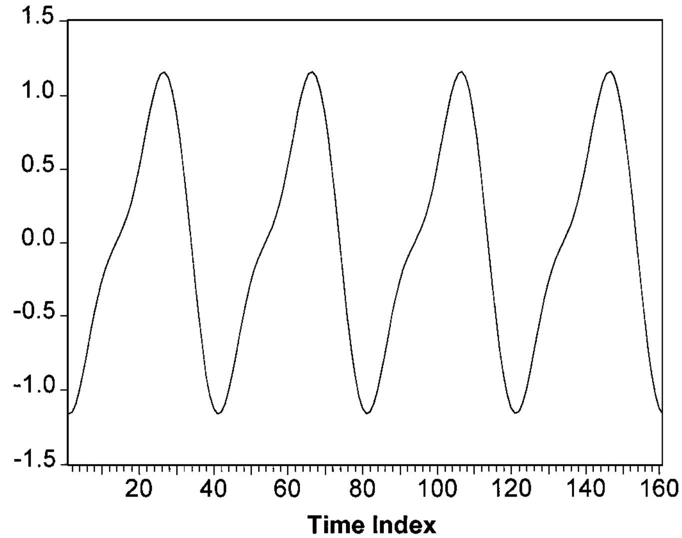

FIGURE 1. Time-series plot of slow up-fast down steep asymmetric cycle.

that shows that business-cycle expansions are not symmetric with contractions. Ramsey and Rothman (1996) argued that the concepts of time reversibility and its inverse, time irreversibility, provide a unified framework for the current alternative definitions of business-cycle asymmetry. The central issue with the detection of time irreversibility in an economic time series is the implication for modeling the series. Time reversibility implies certain symmetries in the formulation of the dynamical equations of motion that are broken in time-irreversible systems. If the nature of time irreversibility can be characterized in parametric terms, this can serve as a guide in how to model the system to meet the dynamical constraints indicated by the presence or absence of time reversibility.

5. TESTING BUSINESS-CYCLE SYMMETRY WITH THE REVERSE TEST

We applied the REVERSE test to a set of three postwar representative monthly macroeconomic time series from the OECD Main Economic Indicators database for five countries: Canada, Germany, Japan, the United States, and the United Kingdom. For each country, we examined the following three business-cycle indicators: the index of industrial production, the consumer price index (CPI), and the aggregate unemployment rate. The sample period for each time series was roughly 1960:01 to 1994:07; a couple series began one year later. We took growth rates of the industrial production and CPI series and raw first differences of the unemployment-rate series before calculating the REVERSE test statistics. The standard tests indicate that these transformations eliminate all evidence of unit root nonstationarity from the original time series.

TABLE 2. Summary statistics and REVERSE test results for growth rates of industrial production indexes

Statistic	Canada	Germany	Japan	United States	United Kingdom
	Country				
Sample period	1961:02–1994:06	1960:02–1994:06	1960:02–1994:07	1960:02–1994:07	1960:02–1994:06
Number of observations	401	413	414	414	413
Mean	0.295	0.223	0.506	0.274	0.149
Standard deviation	1.180	1.818	1.420	0.844	1.560
Skewness	−0.277	0.035	−0.142	−0.696	−0.213
Kurtosis	0.579	7.260	−0.118	2.880	7.630
p-value of REVERSE test statistic	0.122	0.000	0.010	0.006	0.000

TABLE 3. Summary statistics and REVERSE test results for inflation rates

Statistic	Canada	Germany	Japan	United States	United Kingdom
	Country				
Sample period	1960:02–1994:07	1960:02–1994:07	1960:02–1994:07	1960:02–1994:07	1960:02–1994:07
Number of observations	414	414	414	414	414
Mean	0.433	0.305	0.430	0.405	0.597
Standard deviation	0.427	0.404	0.862	0.355	0.706
Skewness	0.431	0.159	0.617	0.457	1.380
Kurtosis	1.510	4.210	1.490	1.100	4.440
p-value of REVERSE test statistic	0.045	0.007	0.283	0.004	0.013

Summary statistics and p-values for the REVERSE test statistics for these transformed series are reported in Tables 2, 3, and 4. For all countries except Canada, time reversibility of the industrial production growth rates is rejected at the 1% significance level, with the strongest rejections occurring for Germany and the United Kingdom. Time reversibility of the CPI inflation rates is rejected at the 5% significance level and lower for all countries except Japan. Likewise, the first-differenced unemployment rates for all countries except Japan appear to be strongly time irreversible.

Table 4. Summary statistics and REVERSE test results for first differences of unemployment rates

	Country				
Statistic	Canada	Germany	Japan	United States	United Kingdom
Sample period	1960:02–1994:07	1962:02–1994:06	1960:02–1994:07	1960:02–1993:12	1960:02–1994:07
Number of observations	414	389	414	407	414
Mean	0.010	0.023	0.002	0.003	0.108
Standard deviation	0.236	0.107	0.096	0.193	0.116
Skewness	0.677	0.595	0.247	0.576	0.128
Kurtosis	0.218	0.585	1.270	1.700	2.420
p-value of REVERSE test statistic	0.027	0.020	0.258	0.010	0.033

The evidence in favor of time irreversibility for these macroeconomic indicators is consistent with longitudinal business-cycle asymmetry as defined by Ramsey and Rothman (1996). The postwar industrial production asymmetry results are particularly interesting, given that DeLong and Summers (1986), Falk (1986), and Verbrugge (1996) failed to find significant asymmetric effects in the dynamical behavior of the output growth rates they examined. This suggests that the REVERSE test may have greater power than alternative tests of business-cycle asymmetry considered in the literature.

6. CONCLUSIONS

We introduce and analyze and REVERSE test, a frequency-domain test of time reversibility. This is the first frequency-domain test of time reversibility available in the statistical time-series literature. Related to Hinich's (1982) bispectrum-based Gaussianity test, our test examines whether the imaginary part of the estimated bispectrum is equal to zero. The REVERSE test signals whether rejection of Gaussianity is due to time irreversibility. Note that non-Gaussian i.i.d. processes lead to rejections of Hinich's Gaussianity test but fail to reject the null hypothesis of the REVERSE test, because all i.i.d. processes are time reversible.

We establish the asymptotic distribution of the REVERSE test statistic. Through Monte-Carlo simulations, we also explore the finite-sample properties of the test. This analysis shows that the test is well behaved for the sample size considered. Thus, it is highly unlikely that rejections of time reversibility are due to size distortions of the test.

In our application of the REVERSE test to the question of business-cycle asymmetry, we established that asymmetry is the rule rather than the exception for a representative set of international monthly macroeconomic time series. The evidence in favor of business-cycle asymmetry is very strong for Germany, the United States, and the United Kingdom, slightly weaker for Canada, and weakest for Japan. The REVERSE test, then, appears to have greater power against time-irreversible, and therefore asymmetric, alternatives than some previous approaches applied in the business-cycle asymmetry literature.

NOTES

1. For example, Lewis et al. (1989) showed that the nonlinear random-coefficient gamma MA(1) process is time reversible because its bivariate characteristic function is symmetric.
2. See Potter (1995) and Koop et al. (1996) for analysis of state-dependent impulse response functions.

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APPENDIX

The proof uses the properties of the joint cumulants of the vector of discrete Fourier transform (DFT) of a section of a sampled time series $\{x(t) : t = 0, 1, \ldots, L-1\}$. The DFT of this data segment for the kth Fourier frequency $f_k = k/L$ is

$$X(k) = \sum_{t=0}^{L-1} x(t) \exp(-i2\pi f_k t). \tag{A.1}$$

Because the $x(t)$'s are real, $X(k) = X^*(-k)$ for each k.

Because cumulants are unfamiliar to most users of statistical theory and methods, let us start with an introduction to cumulants and their relationships with moments. Although most of the results in this Appendix are in the text or exercises in Brillinger (1975) or in papers in the mathematical statistics literature, they are presented here as simply as possible to open the results to people who have a basic understanding of mathematics but who are not able or willing to assemble the pieces on their own with no manual.

Let X denote a random variable whose density function has finite moments of all order, i.e., $\mu_n = E(X^n)$ exists for all integers n. The *moment generating function* (m.g.f) of X is $g_x(s) = E[\exp(sX)]$. Although the generating function is superscripted by x, the notation in this Appendix is simplified if the moments are only subscripted by the order of the moment. The moments and cumulants are subscripted by their random variables when it is necessary to avoid ambiguity.

The natural log of $g_x(s)$ plays an important role in proofs of central limit theorems. The coefficient of $s^n/n!$ in the Taylor-series expansion $\ln[g_x(s)]$ is called the nth *cumulant* of X, and is usually denoted κ_n. An equivalent definition is that κ_n is the nth derivative of $\ln[g_x(s)]$ at $s = 0$. The Taylor-series expansion is

$$\ln[g_x(s)] = \sum_{n=1}^{\nu} \kappa_n (s^n/n!) + o(|s|^{\nu}). \tag{A.2}$$

From the first two derivatives of $\ln[g_x(s)]$ evaluated at $s = 0$, it can be shown that $\kappa_1 = \mu_1 = E(X)$ and $\kappa_2 = \mu_2 - \mu_1^2$. For example, if X has an exponential density $\exp(-x)$, $x \geq 0$, then $\kappa_n = (n-1)!$.

Note that the nth-order cumulant of $X + c$ is also κ_n for $n > 1$. We then can simplify the exposition of cumulants if we set $E(X) = 0$ from now on. Then $\kappa_2 = \mu^2 = \sigma^2$ (the variance of X), $\kappa_3 = \mu_3$, and $\kappa_4 = \mu_4 - 3\sigma^4$. Thus the variance of X^2 is $E(X^4) - \sigma^4 = \kappa_4 + 2\sigma^4$.

The generating function of a vector of ℓ zero-mean random variables $\boldsymbol{X} = (X_1, \ldots, X_\ell)'$ is $g_x(\boldsymbol{s}) = E[\exp(\boldsymbol{s}'\boldsymbol{X})]$, where $\boldsymbol{s} = (s_1, \ldots, s_\ell)'$. For any set of nonnegative integers $n_1, \ldots, n_\ell$ which sum to n, let $\kappa[x_1^{n_1} x_2^{n_2} \cdots x_\ell^{n_\ell}]$ denote the $(n_1, \ldots, n_\ell)n$th order *joint cumulant* of $(X_1^{n_1}, X_2^{n_2}, \ldots, X_\ell^{n_\ell})$. This cumulant is the coefficient of $(s_1^{n_1} \cdots s_\ell^{n_\ell})/(n_1! \cdots n_\ell!)$ in the Taylor-series expansion of $\ln[g_x(\boldsymbol{s})]$. This definition implies that for $n = n_1 + \cdots + n_\ell$,

$$\kappa\left[x_1^{n_1} x_2^{n_2} \cdots x_\ell^{n_\ell}\right] = \frac{\partial^n \ln[g_x(\boldsymbol{s})]}{(\partial s_1)^{n_1} (\partial s_2)^{n_2} \cdots (\partial s_\ell)^{n_\ell}}, \tag{A.3}$$

where the joint partial derivative is evaluated at $\boldsymbol{s} = 0$. The lowercase x's in the cumulant notation should not be confused with the observations $x(t)$ of the time series.

If X is a complex random variable, then its generating function is defined as the bivariate generating function of X and X^*, the complex conjugate of X. In symbols, the generating function of complex X is

$$g_x(t_1, t_2) = E\left[\exp\left(t_1 X + t_2 X^*\right)\right] \quad \text{for real } t_1 \text{ and } t_2. \tag{A.4}$$

This definition of the generating functions of complex variates preserves the algebraic relationships between moments and cumulants that we apply in this paper. The second cumulant of a complex X is $\kappa[x^2] = E(X^2)$. The joint cumulant of (X, X^*) is $\kappa[xx^*] = E(|X|^2)$, which is defined as the *variance* $\operatorname{Var}(X)$ of X.

If $X = (\operatorname{Re} X, \operatorname{Im} X)$, where the real $(\operatorname{Re} X)$ and imaginary $(\operatorname{Im} X)$ terms are independent normal variates whose variance is $\frac{1}{2}$, then X is called a standard complex normal variate. Thus if X is standard complex normal, then $(\operatorname{Re} X)^2$ and $(\operatorname{Im} X)^2$ are independent χ^2 (chi square) variates with one degree of freedom. It follows from the definition of cumulants that the third- and higher-order cumulants of X are all zero.

The generating function for a vector of complex random variables is $g_x(\boldsymbol{t}_1'\boldsymbol{X} + \boldsymbol{t}_2'\boldsymbol{X}^*)$, where $\boldsymbol{X}^*$ is the vector of conjugates of the X_k's. For two complex variates, the joint cumulant $\kappa[x_1 x_2^*] = E(X_1, X_2^*)$ is defined as the *covariance* $\operatorname{Cov}(X_1, X_2)$ of (X_1, X_2).

Note that if $n_\ell = 0$, $s_\ell^{n_\ell} = 1$ and thus in our notation, $\kappa[x_1^{n_1} \cdots x_{\ell-1}^{n_{\ell-1}}]$ is the $(n_1, \ldots, n_{\ell-1})n$th order joint cumulant of $(X_1^{n_1} \cdots X_{\ell-1}^{n_{\ell-1}})$. This implies that the nth cumulant of X_1 is $\kappa_x(x_1^n)$ in our joint cumulant notation. If the X_k's are identically distributed, then $\kappa_x[x^n] = \kappa_n$.

To simplify notation, we now let $\kappa(x_1 \cdots x_\ell)$ denote the $(1, \ldots, 1)$ ℓth-order joint cumulant of the subset $(X_1, \ldots, X_\ell)$ of $\boldsymbol{X}$. When all exponents of the X's are equal to one, the joint cumulant of the X's is called *simple* but the term simple usually is omitted. The third joint cumulant of (X_1, X_2, X_3) is $\kappa[x_1\ x_2\ x_3] = E(X_1, X_2, X_3)$.

For four or more real variates, the relationships between joint moments and cumulant are complicated. For example, the fourth-order joint moment $E(X_1, X_2, X_3, X_4)$ is

$$\kappa[x_1\ x_2\ x_3\ x_4] + \kappa[x_1\ x_2]\kappa[x_3\ x_4] + \kappa[x_1\ x_3]\kappa[x_2\ x_4] + \kappa[x_1\ x_4]\kappa[x_2\ x_3].$$

The following four results are obtained by a methodical application of the definition of the joint cumulant [see Brillinger (1975, p. 19)].

(1) The $n_1, \ldots, n_\ell$ joint cumulant of the permutation $(X_2^{n_2}, X_1^{n_1}, X_3^{n_3}, \ldots, X_\ell^{n_\ell})$ is $\kappa[x_2^{n_2}\ x_1^{n_1}\ x_3^{n_3} \cdots x_\ell^{n_\ell}]$. Thus the simple cumulant of any permutation of $(X_1, \ldots, X_\ell)$ is $\kappa[x_1 \cdots x_\ell]$.
(2) For any constants $a_1, \ldots, a_\ell$, $\kappa[x_1 + a_1 \cdots x_\ell + a_\ell] = \kappa[x_1 \cdots x_\ell]$.
(3) $\kappa[cx_1 \cdots cx_\ell] = c^\ell \kappa[x_1 \cdots x_\ell]$ for any c.
(4) If any subset of $(X_1, \ldots, X_\ell)$ is independent of the other X's, then $\kappa[x_1^{n_1} \cdots x_\ell^{n_\ell}] = 0$.

We now turn to a theorem that is used to develop the joint cumulants of the DFT values $X(k)$ in terms of the joint cumulants $\kappa[x(t_1) \cdots x(t_\ell)]$ of the ℓ observations of the time series.

THEOREM A.1. *Let* $\mathbf{V} = A\mathbf{X}$ *where* A *is a* $K \times \ell$ *matrix whose* k, t *element is denoted* a_{kt}. *The* nth *order joint cumulant of* $(V_{k_1}, \ldots, V_{k_\ell})$ *is*

$$\kappa\left[v_{k_1} \cdots v_{k_n}\right] = \sum_{t_1=1}^{\ell} \sum_{t_2=1}^{\ell} \cdots \sum_{t_n=1}^{\ell} a_{k_1 t_1} a_{k_2 t_2} \cdots a_{k_n t_n} \kappa[x(t_1)x(t_2) \cdots x(t_n)].$$

Proof. To simplify notation, the proof is presented for the case of real a's and X_k's, but the proof holds for complex variates. The m.g.f of $\boldsymbol{V}$ is $g_v(\boldsymbol{s}) = E[\exp(\boldsymbol{s}'\boldsymbol{V})] = E[\exp(\boldsymbol{s}'\boldsymbol{AX})] = g_x(u_1, \ldots, u_l)$, where $\boldsymbol{u} = A's$. Because $\partial u_t/\partial s_k = a_{kt}$, $\partial \ln[g_v/s_k] = a_{k_1}(\partial \ln[g_x]/\partial s_1) + \cdots + a_{kl}(\partial \ln[g_x]/\partial s_l)$ by the chain rule. Thus, $\partial^2 \ln[g_v]/(\partial s_{k_1}\partial s_{k_2}) = \sum_{t_1}\sum_{t_2} a_{k_1}a_{k_2 t_2}(\partial^2 \ln[g_x]/\partial s_{t_1}\partial s_{t_2})$. The result follows by continuing the chain rule for partial differentiation, setting $s = 0$, and applying (A.3). ■

Let $a_{kt} = \exp(-i2\pi f_k t)$, where $t = 0, 1, \ldots, L-1$. Then, from Theorem A.1 and equation (A.1), the joint cumulant of $[X(f_1), \ldots, X(f_n)]$ is

$$\kappa[x(k_1)\cdots x(k_n)] = \sum_{t_1=0}^{L-1}\cdots\sum_{t_n=0}^{L-1} \exp[-i2\pi(f_1 t_1 + \cdots + f_n t_n)]\kappa[x(t_1)\cdots x(t_n)] \quad \textbf{(A.5)}$$

Assume that the time series is strictly stationary. Then, with no loss of generality, assume that $\mu_x = 0$, which implies that $E[X(k)] = 0$ for each x. In addition, the joint cumulant $\kappa[x(t_1)\cdots x(t_n)]$ is a function of the $(n-1)$ lags $\tau_m = t_m - t_n$ $(m = 1, \ldots, n-1)$, which we denote as $\kappa[\tau_1, \ldots, \tau_{n-1}]$.

Assume that the $(n-1)$th-fold sum of $|\kappa[\tau_1, \ldots, \tau_{n-1}]|$ is finite for each n. The nth-fold sum on the right-hand side of expression (A.5) can be written as sums of the $(n-1)\tau$'s and approximated by

$$L\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} W_L(g_1 + \cdots + g_n)S_x(f_1 - g_1, \ldots, f_{n-1} - g_{n-1})dg_1\cdots dg_n, \quad \textbf{(A.6a)}$$

where

$$S_X(\boldsymbol{f}_{n-1}) = \sum_{\tau_1=-\infty}^{\infty}\cdots\sum_{\tau_{n1}=-\infty}^{\infty} \exp[-i2\pi(f_1\tau_1 + \cdots + f_{n-1}\tau_{n-1})\kappa(\tau_1, \ldots, \tau_{n-1})] \quad \textbf{(A.6b)}$$

for the vector $\boldsymbol{f}_{n-1}$ of $(n-1)$ Fourier frequencies $f_1 = k_1/L, \ldots, f_{n-1} = k_{n-1}/L$. Equation (A.6b) is called the $(n-1)$th-order *polyspectrum* [Brillinger (1965)]. The kernel function

$$W_L(\boldsymbol{f}_{n-1}) = \frac{\sin^2(\pi L f_1)\cdots\sin^2(\pi L f_{n-1})}{L^{n-1}\sin^2(\pi f_1)\cdots\sin^2(\pi f_{n-1})} \quad \textbf{(A.6c)}$$

is the $(n-1)$th-fold product of $W_L(f) = L^{-1}\sin^{-2}(\pi f)\sin^2(\pi L f)$, which is called a *Fejer* function in Fourier approximation theory. For large L, the integral (A.6a) is approximately $L\delta(f_1 + \cdots + f_{n-1})S_x(\boldsymbol{f}_{n-1})$ with an error of $\mathcal{O}(1)$ which is proportional to S_x, where $\delta(k) = 1$ if $k = 0$ and $\delta(k) = 0$ if $k \neq 0$. The *spectrum* $S_x(f)$ is the first-order polyspectrum, the *bispectrum* $S_x(f_1, f_2) = B_x(f_1, f_2)$ is the second-order polyspectrum, and the *trispectrum* $S_x(f_1, f_2, f_3) = T_x(f_1, f_2, f_3)$ is the third-order polyspectrum.

Suppose that the observed time-series segment $\{x(1), \ldots, x(N)\}$ of length $N = LP$ is divided into P non-overlapping frames $\{x[(p-1)L + 1], \ldots, x(pL)\}$ of length L. The following theorem is used to obtain the sampling properties for the statistics used in this paper.

THEOREM A.2. *Let $X_p(k) = \sum_n x[(p-1)L + n]\exp(-i2\pi nk/L)$ denote the kth value of the DFT of the pth frame. Assume that $\{x(t)\}$ is strictly stationary and has the following finite memory property: There is a time shift $T \leq L$ such that $x(t)$ and $x(t+T)$*

are independent for each t. Define $Y_p(k_1, k_2) = X_p(k_1)X_p(k_2)X_p(-k_1 - k_2)$ for each p and (k_1, k_2) in the isosceles triangle IT $= \{k_1, k_2 : 0 < k_2 < k_1, k_1 + k_2 < L/2\}$. *There are $M = L^2/16 - L/2 + 1(k_1, k_2)$ in* IT *if L is divisible by 4, approximately $M = L^2/16$ for large L.*

The frame averaged estimate of the bispectrum at $B_x(f_{k_1}, f_{k_2})$ is $\langle Y(k_1, k_2)\rangle/L$, where $\langle Y\rangle$ denotes the arithmetic average of P Y_p's. The ℓ th-*order joint cumulant of* $[\langle Y(k_{11}, k_{12})\rangle, \ldots, \langle Y(k_{\ell 1}, k_{\ell 2})\rangle]$ *is* $P^{-\ell+1}\kappa[y(k_{11}, k_{12}) \cdots y(k_{\ell 1}, k_{\ell 2})][1 + \mathcal{O}(L^{-1})]$.

Proof. Given the finite memory property, a use of Fourier theory similar to that used to obtain the approximation (A.6a) shows that the joint cumulants of any set of $X_p(k_m)$ and $X_{p+1}(k_n)$ is approximately $\mathcal{O}(L)$ if the n integer k's sum to zero, and are $\mathcal{O}(1)$ otherwise. For example, the covariance of $X_p(k_m)$ and $X_{p+1}(k_n)$ is $c_x(1)\exp(\iota 2\pi k/L) + \cdots + (T - 1)c_x(T - 1)\exp[\iota 2\pi(T - 1)/L] = \mathcal{O}(1)$. The joint cumulants of $X_p(k_m)$ and $X_{p+q}(k_n)$ are zero for $q > 1$ because they span two or more frames. The theorem follows from these results and Theorem A.1. ■

The proof of the next theorem uses the major joint cumulants of $Y(k_1, k_2) = X(k_1) X(k_2)X(-k_1, -k_2)$ for various k_1 and k_2.

THEOREM A.3. *Let $Z(k_1, k_2) = P^{1/2}L^{-3/2}\langle Y(k_1, k_2)\rangle$. To simplify notation, suppose that the $\{x(t)\}$ has been prewhitened so that $S_x(f) = \sigma_x^2$ for all f, and set $\sigma_x^2 = 1$. The expected value and variance of $Z(k_1, k_2)$ is $(P/L)^{1/2}B_x(f_1, f_2)[1 + \mathcal{O}(L^{-1})]$ and $1 + \mathcal{O}(L^{-1})$. The second-order joint cumulant of $Z(k_1, k_2)$ and $Z(k_3, k_4)$ is $\mathcal{O}(L^{-1})$ if $k_1 = -k_3$ and $k_2 \neq k_4$, or $k_1 \neq -k_3$ and $k_2 = -k_4$, and is $\mathcal{O}(L^{-2})$ otherwise. Then, the distribution of each $Z(k_1, k_2)$ is approximately a standard complex normal variate in the sense that the mean and variance match the standard normal with an error of $\mathcal{O}(L^{-1})$ and the* ℓ th-*order joint cumulants of the Z's are, at most, $\mathcal{O}(P^{\ell/2+1})[1 + \mathcal{O}(L^{-1})]$.*

Proof. The joint cumulant of a product of variates can be related to the joint cumulants of the variates, but the relationship is complicated. To begin, the combinatorial relationships between the joint cumulants of $X_p(k_1)X_p(k_2)X_p(k_3)$ for various values of k's are developed for a fixed p. The dependence on p is suppressed until needed. The relationships rest on a definition of indecomposable partitions of two-dimensional tables of subscripts of the k's [see Leonov and Shiryaev (1959) and Brillinger (1975, Sec. 2.3)].

Consider the following $\ell \times 3$ table of $k_{j_1}, k_{j_2}, k_{j_3}$, where $k_{j_3} = -k_{j_1} - k_{j_2} (j = 1, \ell)$:

$$\begin{array}{ccc} k_{11} & k_{12} & k_{13} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ k_{\ell_1} & k_{\ell_2} & k_{\ell_3}. \end{array} \tag{A.7}$$

Let $\nu = \nu_1 \cup \cdots \cup \nu_M$ denote a partition of the k_{ji} in this table into M sets, where $j = 1, \ldots, \ell$ and $i = 1, 2, 3$. There are many partitions of the $\ell \times 3$ times from the single set of all the elements to $\ell \times 3$ sets of one element.

The mth set in the partition ν is denoted $\nu_m = (k_{j_{1(m)}i_{1(m)}}, \ldots, k_{j_{\wp(m)}i_{\wp(m)}})$ where $\wp(m)$ is the number of elements in the set. The cumulant of $X[k_{j_{1(m)}i_{1(m)}}), \ldots, X(k_{j_{\wp(m)}i_{\wp(m)}})]$ is $\kappa[x(k_{j_{1(m)}i_{1(m)}}) \cdots x(k_{j_{\wp(m)}i_{\wp(m)}})]$; $\kappa[\nu_m]$ is used for this joint cumulant.

If no two j_i are equal for a set ν_m, then ν_m is called a *chain*. A partition is called *indecomposable* if there is a set with at least one chain going through each row of the table (all rows are chained together). A partition is *decomposable* if one set or a union of some sets in ν equals a subset of the rows of the table.

Consider, for example, the following 2×3 table:

$$\begin{matrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23}. \end{matrix}$$

The decomposable partitions are $(k_{11}, k_{12}, k_{13}) \cup (k_{21}, k_{22}, k_{23})$, which is the union of the two rows and all of its subpartitions. Three indecomposable partitions of this 2×3 table are $(k_{11}, k_{21}) \cup (k_{12}, k_{22}) \cup (k_{13}, k_{23})$, $(k_{11}, k_{22}) \cup (k_{12}, k_{21}) \cup (k_{13}, k_{23})$, $(k_{11}, k_{21}, k_{12}, k_{22}) \cup (k_{13}, k_{23})$. Each pair in these three partitions is a chain.

Let $\nu_r = \nu_1 \cup \cdots \cup \nu_{M_r}$ denote the rth indecomposable partition of table (A.7) into M_r sets. The joint cumulant of $[X(k_{11})X(k_{12})X(k_{13}), \ldots, X(k_{\wp 1})X(k_{\wp 2})X(k_{\wp 3})]$ is the sum over r of the products of the M_r cumulants $\kappa[\nu_m]$ of the ν_m in each indecomposable ν_r.

Recall that $\kappa[x(k_{j_1 i_1}) \cdots x(k_{j_\wp i_\wp})] = \mathcal{O}(1)$, unless $k_{j_i i_1} + \cdots + k_{j_\wp i_\wp} = 0$, when it is $\mathcal{O}(L)$. By an enumeration of each of the cumulants of the sets in the indecomposable partitions of table (A.7) most of the products of cumulants are $\mathcal{O}(1)$ unless the partition consists of sets all of whose indices sum to zero.

The simplest case is for the 2×3 table [$l = 2$ for table (A.7)]. The major second-order joint cumulant of $Y(k_{11}, k_{12})$ and $Y(k_{21}, k_{22})$ for $0 < k_{12} < k_{11}$ and $0 < k_{21} < k_{22}$ in terms of L is

$$\kappa[x(k_1)x(k_2)x(-k_1 - k_2)x(-k_1)x(-k_2)x(k_1 + k_2)] = L^3[1 + \mathcal{O}(1^{-1})], \qquad \textbf{(A.8)}$$

for the partition into three chains $(k_1, -k_1) \cup (k_2, -k_2) \cup (-k_1 - k_2, k_1 + k_2)$. This is the variance of $Y_p(k_1, k_2)$.

Unless $k_{21} = -k_{11}, k_{22} = -k_{12}$ in $(k_{11}, k_{21}) \cup (k_{12}, k_{22}) \cup (k_{13}, k_{23})$, the product is, at most, $\mathcal{O}(L)$. For example, the product for the partition $(k_{11}, -k_{11}) \cup (k_{12}, -k_{22}) \cup (-k_{11} - k_{12}, k_{11} + k_{22})$ when $k_{22} \neq -k_{12}$ is $\mathcal{O}(L)$, which is the joint cumulant $\kappa[y(k_{11}, k_{12})y^*(k_{11}, k_{12})]$.

The product for the partition $(k_1, -k_1, k_2, -k_2) \cup (-k_1 - k_2, k_1 + k_2)$ is $L^2\{T_x(f_1, -f_1, f_2)[1 + \mathcal{O}(L^{-1})]\}$ and similarly for the other partitions into one set of fours and one chain pair where the indices sum to zero.

The cumulant of the whole set is $L[1 + \mathcal{O}(L^{-1})]$. All other products are, at most, $\mathcal{O}(1)$. The second-order cumulants of the Z's are as in the theorem using result 3 and these results.

To obtain the third-order cumulants of the Y_p, one needs to identify the indecomposable partitions of the 3×3 table [$l = 3$ for table (A.7)] where $k_{j3} = -k_{j1} - k_{j2} (j = 1, 3)$:

$$\begin{matrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33}, \end{matrix}$$

which has the major product of cumulants in terms of L. The major product of cumulants holds for the following type of indecomposable partition: $(k_{11}, k_{21}) \cup (k_{12}, k_{22}) \cup (k_{23}, k_{33}) \cup (k_{31}, k_{32}, k_{13})$. If (1) $k_{21} = -k_{11}$, (2) $k_{22} = -k_{12}$, (3) $k_{33} = -k_{23}$, then the major term of $\kappa[|y(k_{11}, k_{12})|^2 y^*(k_{31}, k_{32})]$ is the product $L^4 B_x(f_{31}, f_{32})[1 + \mathcal{O}(L^{-1})]$.

If one of the equalities does not hold, then the product is $\mathcal{O}(L^3)$. There are six indecomposable partitions of the 3×3 table into four pairs and one triple which has a product of cumulants of $\mathcal{O}(L^4)$. The other partitions have products that are $\mathcal{O}(L^j)$ for $j = 1, 2, 3$.

An understanding of the even higher-order joint cumulants of the Y_p is needed to prove the asymptotic properties. The general form can be deduced from the fourth-order case by enumerating the sets in the indecomposable partitions of the 4×3 table [$\mathsf{l} = 4$ for table (A.7)].

Consider the following indecomposable partition:

$$(k_{11}, k_{21}) \cup (k_{12}, k_{22}) \cup (k_{13}, k_{43}) \cup (k_{23}, k_{33}) \cup (k_{31}, k_{41}) \cup (k_{32}, k_{42}).$$

If (1) $k_{21} = -k_{11}$, (2) $k_{22} = -k_{12}$, (3) $k_{43} = -k_{13}$, (4) $k_{33} = -k_{23}$, (5) $k_{41} = -k_{31}$, (6) $k_{42} = -k_{32}$, then the product of the cumulants is $L^6[1 + \mathcal{O}(L^{-1})]$. Because the integers on each row sum to zero, the cumulant $\kappa[|y(k_{11}, k_{12})|^2 |y(k_{21}, k_{22})|^2]$ is $\mathcal{O}(L^6)$.

Note that the six equalities reduce the degrees of freedom of the integers in table (A.7) from eight to four. If any of the equalities are broken, then the joint cumulant of Y's is of lower order in L. Products of indecomposable partitions of the table into six pairs whose k's sum to zero are all $\mathcal{O}(L^6)$.

The pattern for $\mathsf{l} > 4$ is as follows: If $\mathsf{l} = 2n$, the major joint cumulant is of the form $\kappa[|y(k_{11}, k_{12})|^2 \cdots |y(k_{n1}, k_{n2})|^2]$ where the hidden index is $k_{j3} = -k_{j1} - k_{j2}$ for each $j = 1, \ldots, n$. Suppose that for either $i = 1, 2$, or 3, $k_{1i} = k_{2i} = \cdots = k_{ni}$, which constrains the $(k_{11}, k_{12}, \ldots, k_{n1}, k_{n2})$ to an $\mathcal{O}(L^{n+1})$ dimensional subspace of the $\mathcal{O}(L^{2n})$ lattice. Then, the major cumulant is $\mathcal{O}(L^{3n})$. For example, $\kappa[|y(k_{11}, k_{12})|^2 \cdots |y(k_{11}, k_{n2}|^2)$ is $\mathcal{O}(L^{3n})$ if $k_{12} \neq k_{j2}$ for all $j = 1, \ldots, n$.

If this constraint does not hold for the table, the cumulant is $\mathcal{O}(L^{3n-1})$ or smaller. For example, $\kappa[|y(k_{11}, k_{12})|^{2(n-1)} |y(k_{n1}, k_{n2})|^2]$ is $\mathcal{O}(L^{3n-1})$ if $k_{11} \neq k_{n1}$ and $k_{12} \neq k_{n2}$. The variance of $\langle Y_p(k_1, k_2)\rangle$ is $P^{-1}L^{-3}[1 + \mathcal{O}(L^{-1})]$ from (A.8), Theorem A.2 and result 3. The variance of $Z(k_1, k_2)$ is then $1 + \mathcal{O}(L^{-1})$.

Applying these results, for $\mathsf{l} = 2n$ the maximum of the joint cumulants $\kappa[|z(k_{11}, k_{12})|^2 \cdots |z(k_{n1}, k_{n2})|^2]$ is $\kappa[|z(k_{11}, k_{12})|^2 \cdots |z(k_{11}, k_{n2})|^2]$ which is $\mathcal{O}(P^{-n+1})[1 + \mathcal{O}(L^{-1})]$ if $k_{12} \neq k_{j2}$ for all $j = 1, \ldots, n$.

If $\mathsf{l} = 2n + 1$, the major joint cumulant $\kappa[|y(k_{11}, k_{12})|^2 \cdots |y(k_{n1}, k_{n2})|^2 y(k_{\mathsf{l}1}, k_{\mathsf{l}2})]$ is $\mathcal{O}(L^{3n+1})$, which implies that $\kappa[|z(k_{11}, k_{12})|^2 \cdots |z(k_{n1}, k_{n2})|^2 y(k_{\mathsf{l}1}, k_{\mathsf{l}2})]$ is $\mathcal{O}(L^{-1/2} P^{-n+1/2})$ $[1 + \mathcal{O}(L^{-1})]$. Thus the maximum of the lth-order cumulants of $Z(k_1, k_2)$ is $\mathcal{O}(P^{-\mathsf{l}/2+1})$. ■

This result is used in Theorem A.4, which establishes the approximate sampling distribution of our test statistic. Let $\boldsymbol{k} = (k_1, k_2)$ index the two coordinates of the MZ's in IT. The imaginary part of $Z(\boldsymbol{k})$ is $\operatorname{Im} Z(\boldsymbol{k}) = [Z(\boldsymbol{k}) - Z^*(\boldsymbol{k})]/i2$ where $i = (-1)^{1/2}$.

THEOREM A.4. *The distribution of* $2S$ *where* $S = \sum_{\boldsymbol{k} \in \mathrm{IT}} [\operatorname{Im} Z(k)]$ *is approximately* $\chi^2(\lambda)$ *with* $M = L^2/16$ *degrees of freedom and noncentrality parameter* $\lambda = (P/L)^{1/2} \sum_{\boldsymbol{k} \in \mathrm{IT}} [\operatorname{Im} B_x(\boldsymbol{k})]^2$. *For the null hypothesis* H_0: $\operatorname{Im} B_x(\boldsymbol{k}) = 0$ *for all* $\boldsymbol{k}$ *in* IT, *then* S *is approximately a central chi-square variate with* M *degrees-of-freedom for large* L. *A test of size* α *is to reject* H_0 *when* $S > Th$ *where* $\alpha = Pr(S > Th)$. *If* $\operatorname{Im} B_x(k_1, k_2) \neq 0$ *for some* (k_1, k_2), *then the test is consistent if* $P/L \to \infty$ *as* $P \to \infty$.

Proof. Recall that $2[\operatorname{Im} Z(\boldsymbol{k})]^2 = |Z(\boldsymbol{k})|^2 - \operatorname{Re}[Z^2(\boldsymbol{k})]$ and $Z(\boldsymbol{k}) = P^{1/2} L^{-3/2} \langle Y(\boldsymbol{k})\rangle$. $\operatorname{Im} Z(\boldsymbol{k})$ is a normal variate with mean $(P/L)^{1/2} \operatorname{Im} B_x(\boldsymbol{k})\, [1 + \mathcal{O}(L^{-1})]$ and variance $1/2 + \mathcal{O}(L^{-1})$.

The major term of the nth joint cumulant of $[\operatorname{Im} Z(\boldsymbol{k}_1)]^2, \ldots, [\operatorname{Im} Z(\boldsymbol{k}_n)]^2$ depends on $L^{-3n} \kappa[|y(\boldsymbol{k}_1)|^2 \cdots |y(\boldsymbol{k}_n)|^2]$. Enumerating the indecomposable partitions of $n \times 2$ tables of indices of $Y(\boldsymbol{k}_j) Y^*(\boldsymbol{k}_j)$, where $Y(\boldsymbol{k}_j) = X(k_{j_1}) X(k_{j_2}) X^*(k_{j_1} + k_{j_2})$, the major term of this cumulant is the product $L^{-3n}\kappa[(y(k_{11},\ k_{12})y(k_{11},\ k_{22})]\kappa[y(k_{31},\ k_{32})y(k_{31},\ k_{42})]\cdots\kappa[y(k_{(n-1)1},\ k_{n2})]$. This term is $\mathcal{O}(L^{-n})$ because the cumulant is $\mathcal{O}(L^2 n)$. Note that the indices lie on an L^{n+1} dimensional subspace.

The major term of the nth cumulant of $[\mathrm{Im}\, Z(\boldsymbol{k})]^2$ depends on $L^{-3n}\kappa[|y(\boldsymbol{k})|^{2n}]$. There are $(n-1)!$ indecomposable partitions into n pairs of the $n \times 2$ table of indices, where the cumulant of the pair is $\kappa[y(\mathbf{k})y^*(\boldsymbol{k})]$. The major term of the product of these dyad cumulants is $(n-1)!\kappa^{2n}[|y(\boldsymbol{k})|^2]$, which is $(n-1)![1+\mathcal{O}(L^{-1})]$.

If $2S$ is χ^2 with M degrees-of-freedom, then the nth cumulant of S is $M[(n-1)!]$. It now is shown that the nth cumulant of S is $M[(n-1)!] + \mathcal{O}(L^{-1})$. The nth cumulant of S is $M\kappa\{[\mathrm{Im}\, z(\boldsymbol{k})]^2\}$ plus the sum of the nth-order joint cumulants of the $[\mathrm{Im}(\boldsymbol{k}_1)]^2, \ldots, [\mathrm{Im}(\boldsymbol{k}_n)]^2$, where the indices differ. The first term is $[(n-1)!]M[1+\mathcal{O}(L^{-1})]$ from the result in the preceding paragraph, and the maximum of the second term is a sum of $\mathcal{O}(L^{n+1})$ of terms of $\mathcal{O}(L^{-n})$. Thus, this second term is $\mathcal{O}(L)$. Because M is $\mathcal{O}(L^2)$, the error in the nth cumulant is $\mathcal{O}(L^{-1})$. ∎