The Application of the Discrete Fourier Transform in the Estimation of Power Spectra, Coherence, and Bispectra of Geophysical Data

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Abstract. The computation of power spectra, cross spectra, coherence, and bispectra of various types of geophysical random processes is part of the established routine. Since it is routine, some of the standard procedures need to be examined rather carefully to be certain that the assumptions behind the procedures are applicable to the data on hand. The basic criteria for a particular method are its resolution bandwidth, its variance, and its bias. In this paper several basic power-spectrum estimation procedures are reviewed and their statistical and mathematical properties are discussed. The direct use of the discrete Fourier transform for various spectrum calculations is discussed in detail, and its properties are compared with the standard procedure that uses the cosine transform of the estimated correlation function.

INTRODUCTION

The computation of power spectra, cross spectra, and bispectra of geophysical data is part of the established analysis routine [*Tukey,* 1965]. Since it is routine, some of the standard computation procedures need to be examined rather carefully to be certain that the assumptions behind the procedures are applicable to the set of data in question. In measuring the applicability of a procedure, we must determine the following: the resolution bandwidth, the variance, and the bias due to leakage of power from one spectrum region into another.

Until recently most computations have been made by using the correlation-cosine transformation procedure [*Blackman and Tukey,* 1959] Application of the same type of procedure to the measurement of the bispectra promises to be a very lengthy calculation [*Hasselman et al.,* 1963]. Using the fast Fourier transformation permits some of these quantities to be calculated more easily. The direct Fourier transformation is a different type of calculation, and it seems to be particularly suitable for the analysis of microbarograph data.

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A typical finite data sample is shown in Figure 1. Notice that the diurnal component of atmospheric pressure fluctuation is extremely strong. To study shorter sections of data, i.e. to compare power spectra at different times of the day, the effect of the diurnal must be eliminated. For short records the diurnal effect is seen as a slowly varying mean. High-pass filters can be used to reduce the low-frequency components; however, sharp cutoffs require lengthy convolution calculations. Another method is to fit a suitable low-frequency function to the data and then subtract this function from the data. For example, we could fit a sine function to the data sample and determine the frequency, amplitude, and phase of the diurnal component. Over shorter samples we could choose to fit first- or second-degree polynomials. The Fourier transformation is also a least-squares fit to the data, however, and it is aparent that we can discard the low-frequency components and keep the high-frequency components. One calculation then serves both to filter and to analyze the data.

Attractive as the direct transform method is, we must consider very carefully its variance and resolution bandwidth. If we were to make a direct transformation of the whole data sample, the resolution would be very high but the variances of the Fourier components would also be large.

For a given sample of data, the variances of the estimates are generally large when the resolution is high (or the bandwidth is narrow). The trick is to find a procedure that permits us to average a number of independent estimates of the spectrum component and thus to reduce the variance of the estimate. It is important to remember that the averaging procedure must be chosen to match the physical processes (i.e., we cannot improve our estimate of apples by averaging crab apples, peaches, and apples).

The method of spectrum smoothing (Hanning) has been widely used and is applicable to many physical processes. Assume, for example, that we are measuring the pressure fluctuations from a great many independent sources and that each of the sources produces signals that have broad spectrums. The situation is sketched in Figure 2. The geophysical purpose of the spectrum analysis is to attempt to relate sections of the observed spectrum at the receiver to the different sources.

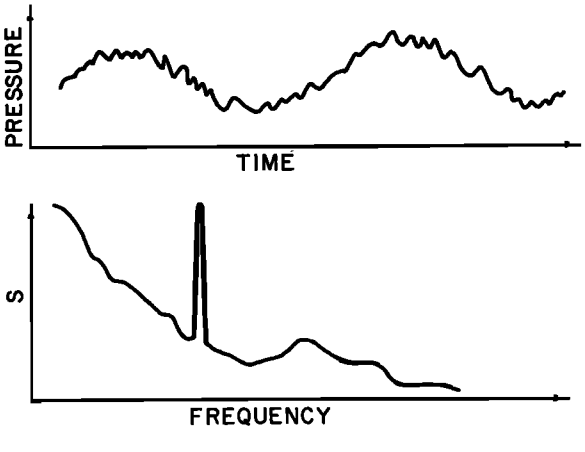
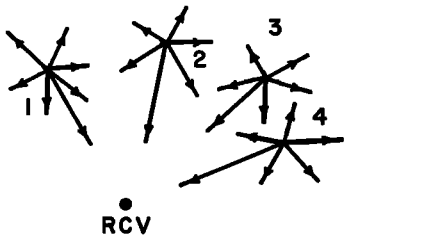


Fig. 1. Power spectrum of a pressure record.



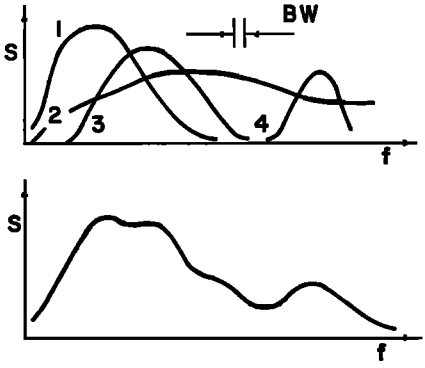


Fig. 2. Spectrum of received signal from four sources. BW is the bandwidth.

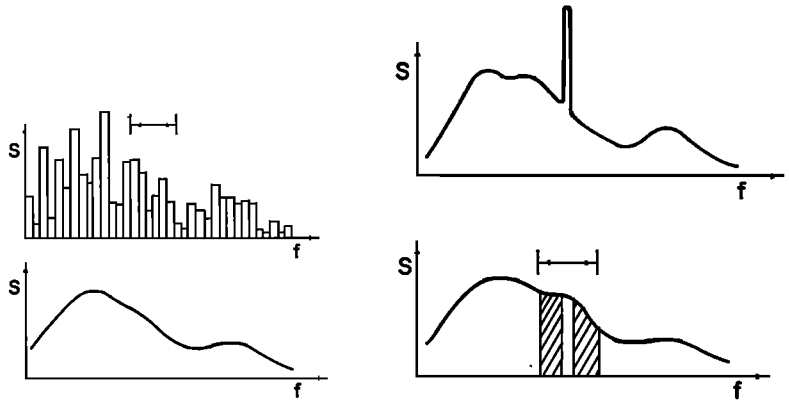


Fig. 3 (left). Spectra before and after smoothing. The bar indicates the width of the smoothing function.

Fig. 4 (right). Spectra having periodic or nearly periodic components: top, actual spectrum; bottom, smoothed spectrum. The bar indicates the width of the smoothing function.

Let us now assume that the spectrum at the receiver has been measured at moderately high resolution. Measurement in each frequency band is indicated in Figure 3. The fluctuations of the spectrum are quite large. Note also (Figure 2) that several frequency bands apply to the same physical process. Thus it seems that we could legitimately reduce the variance of the estimate by averaging the spectrum levels in adjacent frequency bands for any one particular process. A smoothed spectrum is sketched in the lower part of Figure 3.

If the physical processes also include a very strong periodic driving func-

tion such as tides or day-night heating and cooling of the atmosphere, then the spectrum may have the appearance sketched in Figure 4. It is evident that application of the smoothing function broadens the periodic components. The cross-hatching indicates regions where the power due to the periodic components is spread over the power due to the other sources. Spectrum smoothing is unsatisfactory in this situation.

There are alternatives to simple spectrum smoothing, and we shall examine one method rather carefully. Its suitability will be determined on the basis of leakage, large sample variance of the estimate, and bias.

STATISTICAL PROPERTIES

The statistical properties of the direct Fourier transform method for estimating the spectrum of a random process will be discussed and compared with the properties of the Blackman-Tukey procedure. We will not discuss the well-known sampling problem of a signal but will simply assume that the sampling interval is sufficiently small so that the spectrum of the sampled signal is not aliased.

Consider a discrete stationary process $\{X(t\Delta)\}$, where Δ is the time interval between successive obervations of the process. Let $S(f)$ denote the power spectrum of the process. Consider the finite record $X_0, \cdots, X_n$, where $X_t = X(t\Delta)$. *Cooley and Tukey* [1965] published an efficient algorithm, now called the fast Fourier transform, for the computation of the discrete Fourier transform of $X_0, \cdots, X_{n-1}$, i.e., an algorithm for the computation of the n complex Fourier coefficients

$$A_k = (\Delta/n)^{1/2} \sum_{t=0}^{n-1} X_t \exp [2\pi ik(t/n)] \qquad \begin{matrix} k = 0, 1, \cdots, n-1 \\ i = \sqrt{-1} \end{matrix} \tag{1}$$

By taking the inverse transform we have

$$X_t = \frac{1}{(n\Delta)^{1/2}} \sum_{k=0}^{n-1} A_k \exp [-2\pi it(k/n)] \tag{2}$$

To see the connection with the usual form of the discrete Fourier transform, let

$$A(f) = \sum_{t=0}^{n-1} X(t\Delta) \exp (2\pi it\Delta f)$$

Thus

$$A_k = A(f_k)(n/\Delta)^{1/2}$$

where $f_k = k/n\Delta$ Hz. For example, let

$$X(t\Delta) = \exp (-2\pi it\Delta f_0)$$

where $|f_0| < 1/2\Delta$. Then

$$\text{Re } [A(f_k)] = \frac{\sin (2n-1)\pi\Delta(f_k - f_0)}{2 \sin \pi\Delta(f_k - f_0)} + \frac{1}{2}$$

If $f_0 = r/n\Delta$ for some integer r, i.e., if there are exactly r cycles in the record,

$$A(f_k) = \begin{cases} n & \text{if} \quad k = r \\ 0 & \text{if} \quad k \neq r \end{cases}$$

A method will be described that makes direct use of the Fourier transform for the calculation of power spectra. This direct method is several orders of magnitude faster than the Blackman-Tukey approach [*Alsop and Nowroozi,* 1966], owing to the efficiency of this algorithm and to the fact that this method involves n log n operations instead of the n^2 operations used in the older way. The finite discrete Fourier transform is discussed in detail by *Gentleman and Sande* [1966] and *Bingham et al.* [1967].

From equation 1 it follows that $A_k = A_{n-k}^*$, where the asterisk denotes the complex conjugate. Thus $|A_k|^2 = |A_{n-k}|^2$. The numbers $|A_0|^2, \cdots, A_{(n/2)-1}|^2$ are called the *periodogram ordinates* of the sample at the discrete frequencies $f_k = k/n\Delta$, where $k = 0, 1 \cdots, (n/2\Delta) - 1$, and n is even for convenience.

The periodogram ordinates can be expressed in another form. By straightforward algebra

$$|A_k|^2 = \Delta \sum_{\tau=-n+1}^{n-1} \left(1 - \frac{|\tau|}{n}\right) C_\tau \exp\left(-2\pi i \tau \frac{k}{n}\right) \tag{3}$$

where for $\tau \geq 0$

$$C_\tau = \frac{1}{n - \tau} \sum_{t=0}^{n-\tau-1} X_t X_{t+\tau}^* \tag{4}$$

and $C_{-\tau} = C_\tau^*$. Let μ be the mean of the process, i.e. $E(X_t) = \mu$ for each t, and for convenience assume it is real. Let ρ_τ denote the τth covariance term of the process, i.e.

$$\rho_\tau = E(X_t X_{t+\tau}^*) - \mu^2$$

Since by definition of S

$$S(f) = \Delta \sum_{\tau=-\infty}^{\infty} \rho_\tau \exp(-2\pi i \tau \Delta f) \tag{5}$$

taking the expected value of equation 3, it follows that for each k

$$\begin{aligned} E(|A_k|^2) &= \Delta \sum_{\tau=-n+1}^{n-1} \left(1 - \frac{|\tau|}{n}\right)(\rho_\tau + \mu^2) \exp\left(-2\pi i \tau \frac{k}{n}\right) \\ &= \int_{-1/2\Delta}^{1/2\Delta} S\left(\frac{k}{n\Delta} - f\right) \frac{\sin^2 n\pi\Delta f}{n \sin^2 \pi\Delta f}\, df + \begin{cases} \mu^2 n\Delta & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k > 0 \end{cases} \end{aligned} \tag{6}$$

Thus for very large n we have the approximation

$$E(|A_k|^2) \approx \begin{cases} S(f_k) + \mu^2 n\Delta & \text{if} \quad k = 0 \\ S(f_k) & \text{if} \quad k > 0 \end{cases} \tag{7}$$

In many applications the $\mu^2 n\Delta$ term is removed by subtracting the sample

mean

$$\bar{X} = \frac{1}{n} \sum_{t=0}^{n-1} X_t$$

from each observation X_t. Using the discrete Fourier transform, the subtraction of $\bar{X}$ is easily accomplished by setting $A_0 = 0$, since from equations 1 and 2, $\bar{X} = 0$ if and only if $A_0 = 0$.

For a finite n, the expected value of $|A_k|^2$ can be interpreted as the power in a band around $f_k = k/n\Delta$ of approximate bandwidth $1/n\Delta$ Hz. However, if the power spectrum has some large peaks near f_k or if the spectrum has a large slope around f_k, the side lobes of filter function $(\sin^2 n\pi\Delta f)/(n \sin^2 \pi\Delta f)$ (called the Fejér kernel) will 'leak' power from the adjacent bands to such an extent as to produce a severe bias for $|A_k|^2$; i.e., $|A_k|^2$ will not be a good estimate of the power in the band around f_k owing to its systematic error. After we have examined the statistical properties of the $|A_k|^2$, we shall return to this problem.

It has been shown [*Bartlett,* 1955] that under some fairly weak conditions for the random process, such as that the process is linear but not necessarily Gaussian, the periodogram ordinates $|A_k|^2$ for large n are approximately independent random variables. Their large-sample variances are given by

$$\text{var}\,(|A_k|^2) \approx \begin{cases} 2S^2(f_k) & \text{if} \quad k = 0 \\ S^2(f_k) & \text{if} \quad k > 0 \end{cases} \tag{8}$$

Moreover for large n, $2|A_k|^2/S(k/n\Delta)$ have approximately a *chi-squared distribution* with 2 degrees of freedom for $k \geq 1$ and with 1 degree of freedom for the term $|A_0|^2/S(0)$.

Since the variance of $|A_k|^2$ does not go to zero as $n \to \infty$, the periodogram ordinates are not consistent estimates of the power spectrum. Furthermore, the asymptotic independence of $|A_k|^2$ and $|A_j|^2$ implies that from sample to sample (non-overlapping) from the same random process the periodogram fluctuations appear highly erratic when plotted against k. The standard deviation of $|A_k|^2$ is 100% of its mean, which implies that for any given k the observed value of $|A_k|^2$ can lie anywhere from 0 to $2\,S(k/n\Delta)$. The periodogram peaks will include many false peaks, owing to sampling fluctuations. The inconsistency of the periodogram forced investigators to seek other means of computing spectra, such as computing the main part of the autocovariance function and then taking its Fourier transform [*Parzen,* 1957]. To obtain a meaningful estimate of the power spectrum with a given sample size of n, the $1/n\Delta$—Hz resolution of the periodogram must be reduced to reduce the fluctuations of the estimate.

A simple method of trading resolution for smaller variance is to break up the record into non-overlapping pieces and then to compute the average value for each of the corresponding periodogram ordinates. To be more explicit, if r is a divisor of n, the record is broken into r pieces, each of which has m points, where $m = n/r$. The pieces are m points apart. Then the m numbers

$$S_k = \frac{1}{r} \sum_{p=1}^{r} |A_k^{(p)}|^2 \qquad k = 0, \cdots, m/2 - 1 \tag{9}$$

are computed, where for each $p = 1, \cdots, r$,

$$|A_k^{(p)}|^2 = \frac{\Delta}{m}\left|\sum_{t=0}^{m-1} X_{t+(p-1)m} \exp(2\pi i t)(k/m)\right|^2 \tag{10}$$

This averaging technique is discussed by *Haubrich* [1965] for use in cross-spectrum, coherency, and bispectrum analysis, as well as for power-spectrum calculations. Haubrich did not, however, give the large sample variances and bias of the estimators; instead, Monte Carlo simulation was used to estimate the variances of these estimators for a certain time-series model. Further discussion of statistical properties of the direct use of the Fourier transform is given by *Hannan* [1960] and [*Welch* [1961]. From equation 6 the expected value of S_k is

$$E(S_k) = \int_{-1/2\Delta}^{1/2\Delta} S\left(\frac{k}{m\Delta} - f\right) \frac{\sin^2 m\pi\Delta f}{m \sin^2 \pi\Delta f}\, df \tag{11}$$

Thus for large m

$$E(S_k) \approx S(f_k), \qquad f_k = k/m\Delta$$

Thus the S_k give asymptotically unbiased estimates of the power spectrum evaluated at frequencies $1/m\Delta$ Hz apart. For finite n, S_k is a good estimate of the power in the band around $f_k = k/m\Delta$ of approximate bandwidth $1/m\Delta$ Hz. As mentioned earlier, the bias of the estimator S_k will be small if the power spectrum is fairly flat for a few bands about f_k. When data are analyzed in this way, it is important to remember that several different values of r (and thus m) should be tried if it is believed that there are sinusoidal components in the data, i.e., if there are spectral lines in the true spectrum.

If m is larger than the correlation distance of the random process ($1/m\Delta$ is smaller than the bandwidth), then $|A_k^{(p)}|^2$ and $|A_k^{(s)}|^2$ are approximately uncorrelated for $p \neq s$. Thus from equations 8 and 9, for large m the variance of S_k is approximately

$$\begin{aligned} \text{var}\,(S_k) &\approx \frac{1}{r^2}\sum_{p=1}^{r} \text{var}\,(|A_k^{(p)}|^2) \\ &\approx \frac{m}{n} S^2(f_k) \end{aligned} \tag{12}$$

where $1/r = m/n$.

Another method involving the direct use of the discrete Fourier transform is discussed by *Jones* [1965]. In this method, first suggested by *Daniell* [1946], the n periodogram ordinates are calculated from the whole record, and then blocks of $r = n/m$ ordinates are averaged to give $m/2 = n/2r$ estimates of the spectrum up to the folding frequency. The averaging or smoothing produces the required trade-off between resolution and variance. The large-sample proportional variance is reduced from 1 to $1/r = m/n$, and the resolution is reduced from $1/n\Delta$ *to* $1/m\Delta$ Hz. This method gives the same large-sample variance and resolution as the method cited above. However, if $S(f)$ has one or more large peaks in a band of interest (of bandwidth $1/m\Delta$), the smoothing causes a greater

bias in the estimated value of the spectrum than is produced by the previously described method involving the Fejér kernel. On the other hand, if the true spectrum is slowly varying over the frequency bands of interest, periodogram smoothing is a simple and effective way of reducing the variance of the periodogram type of power-spectrum estimators.

Let us now compare the above procedure with the Blackman-Tukey method. The main step in their method is to compute the Fourier transform of the first M covariance terms, i.e., to compute for $k = 0, \cdots, M$, the raw power-spectrum estimates

$$U_k = \Delta \sum_{\tau=-M}^{M} C_\tau \exp\left(-\pi i \tau \frac{k}{M}\right) \tag{13}$$

where $M \ll n$ and where the sample mean $\bar{X}$ has been subtracted from the data.

In order to properly compare the methods, let $M = m/2$, where $m = n/r$. For finite n, the expected value of U_k is

$$\begin{aligned} E(U_k) &= \Delta \sum_{\tau=-M}^{M} \rho_\tau \exp\left(-2\pi i \tau \frac{k}{m}\right) \\ &= \int_{-1/2\Delta}^{1/2\Delta} S(f_k - f) \frac{\sin (m + 1)\pi\Delta f}{\sin \pi\Delta f} \, df \end{aligned} \tag{14}$$

and thus for large m

$$E(U_k) \approx S(f_k) \tag{15}$$

where $f_k = k/m\Delta$ Hz. It can be shown [*Bartlett*, 1955] that, for large m, U_k and U_l are approximately independent if $k \neq l$, and also

$$\text{var } (U_k) \approx (m/n) S^2(f_k) \tag{16}$$

Thus both U_k and S_k give asymptotically (as $m \to \infty$) unbiased estimates of the power spectrum $S(f)$, with identical large-sample variances $m/n\ S^2(f_k)$.

For finite m, U_k is also a good estimate of the power in the band about f_k of bandwidth $1/m\Delta$ Hz, provided $S(f)$ is smooth and does not have large peaks. As indicated by equation 14, if $S(f)$ has a large peak at a frequency f_0 (due to a sinusoidal component in the data with period $1/f_0$), the major side lobes of the Dirichlet kernel $[\sin\ (m + 1)\ \pi\Delta f]/(\sin\ \pi\Delta f)$ leak power and cause considerable bias in the estimates of the power in adjacent bands of f_0. The leakage problem for several important kernels will be discussed in more detail in the next section. It is clear, however, that the side-lobe area of the Dirichlet kernel is greater than that of the Fejér kernel $(\sin^2 m\pi\Delta f)/(m \sin^2 \pi\Delta f)$, since the Dirichlet kernal decays as $O(f^{-1})$, whereas the Fejér kernel decays as $O(f^{-2})$. Thus the bias due to side-lobe leakage of the raw spectrum estimates U_k is greater than the bias of S_k. Furthermore, since

$$\sum_{\tau=-M}^{M} C_\tau \neq 0$$

even if $\mu = 0$, the zero-frequency estimate U_0 can be large in contrast with S_0.

There exists, however, a very simple method to decrease both the variance and the side-lobe leakage of the U_k estimates, provided that the true spectrum is not too convoluted. Consider the following smoothing of the U_k:

$$V_k = c_{-1}U_{k-1} + c_0U_k + c_1U_{k+1}$$

where $\sum c_j = 1$. If $c_0 = \frac{1}{2}$ and $c_1 = c_{-1} = \frac{1}{4}$, this smoothing operation is called Hanning. Since $\sum c_j = 1$ and the U_k are independent for large m, it follows from equations 14, 15, and 16 that

$$\begin{aligned} E(V_k) &= \int_{-1/2\Delta}^{1/2\Delta} S(f_k - f)K(f)\,df \\ &\approx S(f_k) \end{aligned} \tag{17}$$

where

$$K(f) = \sum_{j=-1}^{1} c_j \frac{\sin (m+1)\pi\Delta[f + (j/m\Delta)]}{\sin \pi\Delta[f + (j/m\Delta)]} \tag{18}$$

and

$$\text{var}\,(V_k) \approx (c_{-1}^2 + c_0^2 + c_1^2)(m/n)S^2(f_k) \tag{19}$$

For Hanning smoothing, the large-sample variance of the V_k is $(3M/4n)\ S^2(f_k)$, where $M = m/2$ is the number of lags used. Thus Hanning reduces the variance of the spectral estimates by 3/8.

The bandwidth of the Hanning kernel (K with $c_0 = 1/2$, $c_1 = c_{-1} = 1/4$) is greater than that of the Dirichlet kernal, as seen in Figure 5 with $\Delta = 1$ sec. If at a given frequency, f_k, the true spectrum has peaks at the adjacent frequencies f_{k+1} and f_{k-1}, the Hanning smoothing will take power from these adjacent bands and will give an overestimate of the power at f_k. The twin peaks

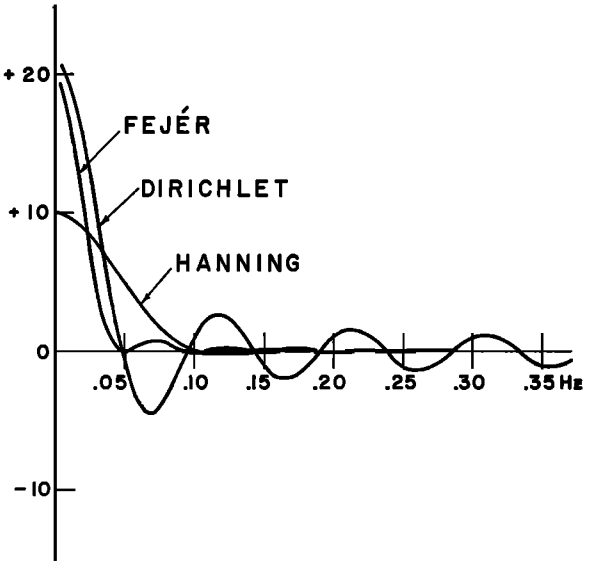


Fig. 5. Three spectral kernels with $m = 20$.

will be blurred into one hump, whereas the unsmoothed U_k would show the two peaks. Since the major side lobes of the Hanning kernel (at $f = 3/2m\Delta$) are smaller than the Dirichlet kernel, they have less leakage from peaks at frequencies farther away from f_k than $1/m\Delta$.

The Fejér kernel has even smaller bandwidth and even lower major side lobes than the Hanning kernel. Consequently, the S_k estimates have less bias than the V_k. However, the large-sample variance of V_k is 3/8 of the variance of S_k. If the true spectrum has many peaks, the smaller variance of the Hanned estimates V_k does not compensate for its larger bias. Furthermore, if the Fourier coefficients $A_k^{(p)}$ are Hanned for each piece $p = 1, \cdots, r$, then the averaged periodogram ordinates (cf. equation 9) will have the same variance as the V_k. The bias will be less, however, since the appropriate kernel for these spectral estimates will be K^2, which has smaller bandwidth than K.

TRENDS AND PERIODIC TERMS

The signal may consist of many small high-frequency components superimposed on a large low-frequency component (see Figure 1). If we observe only a short section of this record, the data would appear to be a fluctuating process superimposed on a drifting or slowly varying mean. In statistical terminology, this slowly varying mean is referred to as a trend.

Let us assume that we are interested in the higher-frequency components and would like to eliminate the low-frequency effect. In order to illustrate the problem of a trend, let us consider the process $\{Y_t\}$, where

$$Y_t = m_t + \sum_{s=1}^{N} a_s \exp(2\pi i t f_s) + X_t \tag{20}$$

with the X_t as discussed above, with $\Delta = 1$ sec and $f_N < \frac{1}{2}$Hz. The a_s coefficients can be complex. In polar form, $a_s = |a_s| \exp[i\phi(f_s)]$, where $\phi(f_s)$ is the phase of the sinusoidal. The term m_t, a low-order polynomial in t, is called the trend. For many models, the trend is linear, i.e., $m_t = \gamma t$. The trend contributes mostly low-frequency power to S_y, the spectrum of $\{Y_t\}$. Thus, to control the bias due to low-frequency leakage in the estimation of S_y, the trend must be filtered out. *Durbin* [1962] discusses the basic problems associated with trend elimination. A moving average high-pass filter (such as a first-difference filter) is usually used to remove the trend, and then the estimated spectrum is corrected for the effect of filtering by dividing it by the filter function. However, the reduction of the leakage bias from the trend depends *only* on the beginning and end of the record, not on the middle points. A good method to fit a trend of known functional form (such as an nth-order polynomial) is to use 'least squares.' The properties of least-squares trend estimation are reviewed by *Hannan* [1960], and they are discussed in detail by *Grenander and Rosenblatt* [1957].

Let us now give the contribution of the periodic terms to the estimate of S_y. For simplicity let us find the contribution of just one of the sinusoidal terms in equation 20 to the expected value of the various spectral estimates. For a given f_s, let the phase $\phi(f_s)$ be chosen from the uniform density with the range $0 <$

$\phi(f_s) < 2\pi$. It then follows that $\{a_s \exp(2\pi i t f_s)\}$ is a stationary random process, whose ensemble is characterized by the ensemble of phase angles, and that its jth covariance term is

$$\rho_\tau = |a_s|^2 \exp(2\pi i \tau f_s)$$

Thus from equation 5, $\{a_s \exp(2\pi i t f_s)\}$ has the power spectrum

$$S(f) = |a_s|^2 \sum_{\tau=-\infty}^{\infty} \exp[2\pi i \tau (f_s - f)] = \delta(f - f_s) \tag{21}$$

where $\delta(f)$ is the Dirac delta function. By applying equation 21 to equations 11, 14, and 17, we have for $f_k = k/n\Delta$

$$E(S_k) = |a_s|^2 \frac{\sin^2 m\pi(f_k - f_s)}{m \sin^2 \pi(f_k - f_s)} \qquad \text{Fejér} \tag{22a}$$

$$E(U_k) = |a_s|^2 \frac{\sin (m+1)\pi(f_k - f_s)}{\sin \pi(f_k - f_s)} \qquad \text{Dirichlet} \tag{22b}$$

$$E(V_k) = |a_s|^2 K(f_k - f_s) \qquad \text{Hanning} \tag{22c}$$

where $K(f)$ is given by equation 18 with $c_0 = \frac{1}{2}$ and $c_1 = c_{-1} = \frac{1}{4}$. Thus as $m \to \infty$, S_k and U_k have expected values that converge to the true power $m|a_s|^2$ at f_s, whereas $E(V_k) \to (m/2)\ |a_s|^2$. At another frequency $f_l \neq f_s$, all the estimators have a bias due to the 'leakage' of power by the side lobes of the kernels (assuming that $f_s \neq k/m\Delta$ for some k).

In order to compare the leakage bias that results in the use of the Blackman-Tukey method (with and without Hanning) with the bias that results in the use of the periodogram method when the data have a substantial nonstationary mean, the nonrandom part of $\{Y_t\}$ was computed for $n = 512$, $m = 128$ (64 lags), and $r = 8$ (pieces). A linear trend, $m_t = \gamma t$, was used with $\gamma = 2/n$, and the frequencies slected were $f_1 = 8.25/512$ Hz plus $f_2 = 16.5/512$ Hz. The periodic components were chosen to be *real* sine functions, and the amplitudes a_s used were $\sqrt{0.5}$ for both frequencies. As an alternate case, the linear trend alone was used as the input for both procedures. The contribution to the power 'estimate' of each of the sine functions was $m/4 = 16$ at the positive frequencies f_1 and f_2.

With the periodogram methods the bias of the power 'estimate' at the ith band is not simply the square of the absolute value of the transform of the trend (with the mean subtracted). There is an additional bias, which is the sum of the products of the real and imaginary parts of the transform of the sinusoidal and the transform of the trend, since for complex numbers z_1 and z_2

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2(\text{Re}\, z_1 \times \text{Re}\, z_2 + \text{Im}\, z_1 \times \text{Im}\, z_2)$$

The contribution using the periodogram method due to the trend alone is 0.2 for f_1 and 0.05 for f_2 (Figure 6), whereas the total bias with the sinusoidals present (Figure 7) is $19.8 - 16 = 3.8$ for f_1 and $16.8 - 16 = 0.8$ for f_2. Thus, although the power contribution of the trend is small, the cross terms give a bias proportional to the power of the sinusoidal (Figure 8).

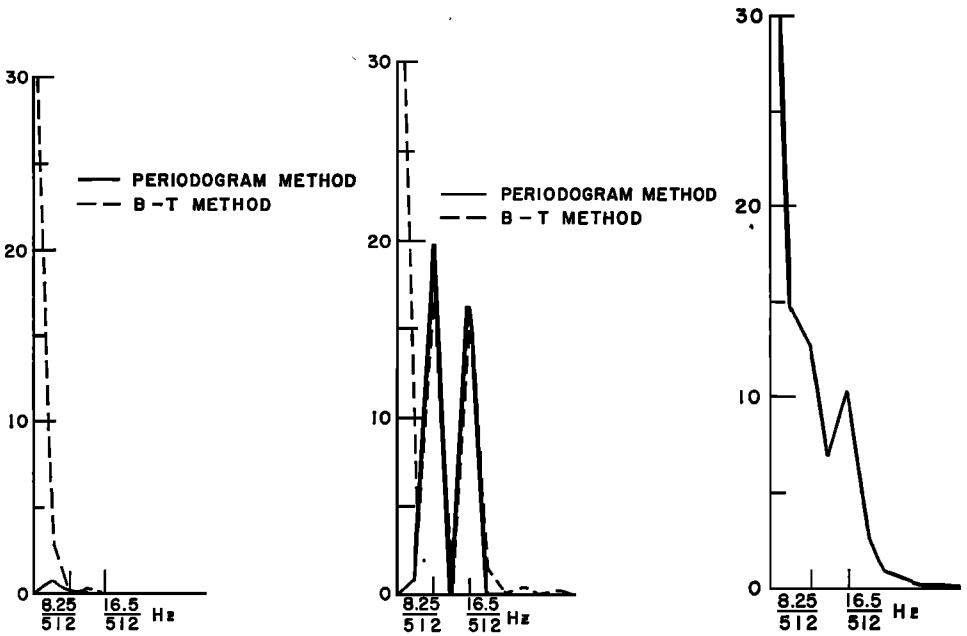


Fig. 6 (left). Spectrum of trend (zero mean) before Hanning.
Fig. 7 (middle). Spectrum of trend and equal sinusoidals before Hanning.
Fig. 8. (right). Spectrum of trend and equal sinusoidals after Hanning for B-T method.

CROSS SPECTRA AND COHERENCE

Many studies of geophysical processes involve the computation of the cross spectra and coherence between signals observed at various sensors in an array. As dictated by array theory, the estimated cross spectra are used to improve the signal-to-noise gain of the array [*Backus et al.*, 1964; *Burg*, 1964]. The use of cross spectra is well known, but it is also well known that the significance of the measurements is often difficult to judge.

Consider two discrete stationary random processes $\{X_t\}$ and $\{Y_t\}$. The *cross spectrum* $S_{xy}(f)$ for this pair of processes is defined by

$$S_{xy}(f) = \Delta \sum_{\tau=-\infty}^{\infty} \rho_\tau^{xy} \exp(-2\pi i \tau \Delta f) \tag{23}$$

where the τth *cross covariance* is

$$\rho_\tau^{xy} = E(X_t Y_{t+\tau}^*) - \mu_x \mu_y^* \tag{24}$$

The *coefficient of coherence* is defined to be

$$\gamma_{xy}(f) = \frac{|S_{xy}(f)|}{[S_x(f) S_y(f)]^{1/2}} \tag{25}$$

where S_x and S_y are the power spectra of $\{X_t\}$ and $\{Y_t\}$, respectively. From the Schwarz inequality, it follows that $0 \leq \gamma_{xy}(f) \leq 1$. The *average phase difference*

is defined to be

$$\phi(f) = \tan^{-1} \frac{\text{Im } [S_{xy}(f)]}{\text{Re } [S_{xy}(f)]} \tag{26}$$

where Re (z) and Im (z) are the real and imaginary parts of a complex z. Thus

$$S_{xy}(f) = \gamma_{xy}(S_x S_y)^{1/2} \exp(i\phi)$$

The coherence coefficient $\gamma_{xy}(f_0)$ is inversely related to the variance of the observed phase difference of the f_0 frequency components of the two processes. That is, if $\gamma_{xy}(f_0) = 1$, then the f_0 components are perfectly in phase with phase difference $\phi(f_0)$. If $\gamma_{xy}(f_0) = 0$, the phase difference of the f_0 components of the two processes has a large variance, and thus the average difference $\phi(f_0)$ does not make any physical sense. The two processes are said to be *incoherent* at f_0 if $\gamma_{xy}(f_0) = 0$.

The statistical problems involved in the estimation of γ_{xy} from finite records of $\{X_t\}$ and $\{Y_t\}$ are not trivial. The first basic work was done by *Goodman* [1957]. Further work in the estimation problem is given by *Shapiro* [1963], *Tick* [1966], and *Foster and Guinzy* [1967]. We will now sketch a method for estimating γ_{xy} using the discrete Fourier transform. This method is similar to the one discussed by *Haubrich* [1965], but we will also state the large-sample variances.

Suppose that equal size records are taken of $\{X_t\}$ and $\{Y_t\}$ and that these records are subdivided into r pieces, each piece containing m points as before. Given the pth pieces $X_{(p-1)m}, \cdots, X_{pm-1}$ and $Y_{(p-1)m}, \cdots, Y_{pm-1}$, define $A_k^{(p)}$ for $k = 0, 1, \cdots, m-1$, as before and

$$B_k^{(p)} = (\Delta/m)^{1/2} \sum_{t=0}^{m-1} Y_{t+(p-1)m} \exp [2\pi i t(k/m)] \tag{27}$$

Then

$$E(A_k B_k^*) = \int_{-1/2\Delta}^{1/2\Delta} S_{xy}(f_k - f) \frac{\sin^2 m\pi\Delta f}{m \sin^2 \pi\Delta f} df \tag{28}$$

which is similar to equation 11. Thus for large m, $A_k B_k^*$ is a good estimate of $S_{xy}(f_k)$ if the cross spectrum is reasonably behaved near f_k. Analogous to the one-dimensional case, consider the estimator

$$S_k^{xy} = \frac{1}{r} \sum_{p=1}^{r} A_k^{(p)} B_k^{*(p)} \tag{29}$$

From equation 28, for large m

$$E(S_k^{xy}) \approx S_{xy}(f_k)$$

The coefficient of coherency and the average phase difference are of primary interest in most problems, however. Consider the following estimate of $\gamma_{xy}(f_k)$:

$$\hat{\gamma}_{xy}(f_k) = |S_k^{xy}|/(S_k^x S_k^y)^{1/2} \tag{30}$$

where S_k^x is the estimate of $S_x(f_k)$ as defined by equation 9, and, similarly, S_k^y

is the estimate of $S_y(f_k)$. By the Schwarz inequality for sums it follows that $0 \leq \hat{\gamma}_{xy} \leq 1$. For large m

$$\hat{\gamma}_{xy} \approx \gamma_{xy}(f_k) \tag{31}$$

and following the technique given by *Jenkins* [1963], it can be shown that

$$\text{var } (\hat{\gamma}_{xy}) \approx (m/2n)[1 - \gamma_{xy}^2(f_k)]^2 \tag{32}$$

Now consider the following estimate of $\phi(f_k)$:

$$\hat{\phi}(f_k) = \tan^{-1} [\text{Im } (S_k^{xy})/\text{Re } (S_k^{xy})] \tag{33}$$

For large m, $\hat{\phi}(f_k) \approx \phi(f_k)$, and the large sample variance is

$$\text{var } (\hat{\phi}) \approx (m/2n)[\gamma_{xy}^{-2}(f_k) - 1] \tag{34}$$

Thus as the coherency $\gamma_{xy} \to 0$, the variance of $_\phi \to \infty$. This is to be expected from the inverse relation between the coherence and the variability of the phase difference. It can also be shown that

$$\text{var } (|S_k^{xy}|) \approx (m/2n)S_x(f_k)S_y(f_k)[1 + \gamma_{xy}^2(f_k)] \tag{35}$$

The estimators $\hat{\gamma}_{xy}$ and $\hat{\phi}$ will give good results for reasonable m and n if the cross spectrum is slowly varying in the frequency bands of interest and if S_x and S_y are not small for these frequencies. Moreover these estimators will also give good results if there is a sinusoidal component in $\{X_t\}$ at $f = f_0$ that is in phase with a sinusoidal component of frequency f_0 in $\{Y_t\}$, provided m is chosen so that there is an integer k such that $f_k = k/m\ \Delta$ is near f_0.

BISPECTRA

Suppose that a zero mean Gaussian process $\{X_t\}$ is passed through a nonlinear filter (such as a squarer). The output of the nonlinear filter is no longer Gaussian. If there are strong sinusoidal components in $\{X_t\}$ with frequencies f_1 and f_2 and phases $\phi(f_1)$ and $\phi(f_2)$, then, owing to the nonlinearity, the component at $f = f_1 + f_2$ will have a phase angle

$$\phi(f_1 + f_2) = \phi(f_1) + \phi(f_2) - \theta \tag{36}$$

where θ is slowly varying over time. The phases of the component of $\{X_t\}$ for $f = f_1 - f_2$ and $f = f_2 - f_1$ will also be *coherent* with $\phi(f_1) + \phi(f_2)$. This could correspond to a physical process in which a wave is propagating in a medium whose response is nonlinear.

The *bispectrum* $B(f_1,f_2)$ gives a measure of the multiplicative nonlinear interaction of frequency component in $\{X_t\}$ [*Hasselman et al.*, 1963; The bispectrum is defined by *MacDonald*, 1963].

$$B(f_1, f_2) = \sum_\sigma \sum_\tau \rho_{\sigma,\tau} \exp [-2\pi i(f_1\sigma + f_2\tau)] \tag{37}$$

where

$$\rho_{\sigma,\tau} = E(X_t X_{t+\sigma} X_{t+\tau})$$

If $\{X_t\}$ is a real process, then

$$B(f_1, f_2) = B(f_2, f_1) = B^*(-f_1, -f_2)$$

If $\{X_t\}$ is a *Gaussian process,* its bispectrum $B(f_1,f_2) \equiv 0$, since in the Gaussian case $\rho_{\sigma,\tau}$ is zero for each σ and τ. *Thus if the bispectrum is not identically zero, the process is not Gaussian.*

Expressed in polar form

$$B(f_1, f_2) = [S(f_1)S(f_2)S(f_1 + f_2)]^{1/2}\rho \exp(i\theta) \tag{38}$$

where $\rho(f_1, f_2)$ is called the skewness and θ (f_1, f_2) is the phase lag. If the component at $f = f_1 + f_2$ is basically due to a multiplicative interaction between the components of $\{X_t\}$ at f_1 and f_2, then $\rho = 1$ and θ is as given in equation 36.

The estimation of the bispectrum from a finite record of the process can be easily computed by the use of the discrete Fourier transform. Consider the following estimate of $B(f_j, f_k)$ for $0 < k < j < m/2$:

$$\hat{B}(f_j, f_k) = \frac{1}{r}\sum_{p=1}^{r} A_j^{(p)} A_k^{(p)} A_{j+k}^{*(p)} \tag{39}$$

where the $A_k^{(p)}$ are given by equation 10. As the estimate of skewness, let

$$\hat{\rho}(f_j, f_k) = \frac{|\hat{B}(f_j, f_k)|}{(S_j S_k S_{j+k})^{1/2}} \tag{40}$$

Using the work of *Rosenblatt and Van Ness* [1965], for large m and n

$$\hat{B}(f_j, f_k) \approx B(f_j, f_k)$$

and

$$\text{var (Re } \hat{B}) = \text{var (Im } \hat{B}) \approx (m/2n)S(f_j)S(f_k)S(f_j + f_k) \tag{41}$$

where Re $\hat{B}$ and Im $\hat{B}$ are independent. Thus if $S(f_j)$, $S(f_k)$, and $S(f_j + f_k)$ are known with certainty, the large sample variance of $\hat{\rho}$ is $m/2n$. However, if the record of $\{X_t\}$ is used to estimate the power spectrum terms as in equation 40, the large sample variance of $\hat{\rho}$ is not trivial to compute.

In many applications, however, the nonlinearity in the process is due to first-order multiplicative interactions between major components of the process (such as between tidal and diurnal cycles). In these cases it is often reasonable to assume that the amplitude value $|A_{j+k}|$ is proportional to $|A_j|\ |A_k|$ for interacting components of frequencies f_j and f_k, and thus the nonzero value of the skewness is mainly due to phase coherence between the f_j, f_k, and f_{j+k} components. Then the large-sample variance of $\hat{\rho}$ is approximately

$$\text{var } (\hat{\rho}) \approx (m/2n)[1 - \rho^2(f_j, f_k)]^2 \tag{42}$$

for each j and k. This result follows by analogy from equation 32, since the bispectrum estimate $\hat{B}$ for $(f_j, f - f_j)$ can be expressed as the appropriate estimate of the *cross spectrum* of $\{X_t\}$ and a process $\{Z_t(f)\}$ as follows:

$$\hat{B}(f_j, f - f_j) = S_j^{x,z(f)} \tag{43}$$

where

$$Z_t(f) = \sum_{s=0}^{n} X_t X_{t-s} \exp(2\pi i s f)$$

since the jth Fourier coefficient of $\{Z_t(f)\}$ is $A(f_j)A(f - f_j)$. In order to have $|\hat{\rho}| \leq 1$, however, the denominator in equation 40 should be

$$\frac{1}{r}\left[\sum_{p=1}^{r} |A_j^{(p)}|^2 \sum_{p=1}^{r} |A_k^{*(p)} A_{j+k}^{(p)}|^2\right]^{1/2}$$

As a simple rule of thumb, ρ should only be estimated when there is a reasonable amount of power at f_j, f_k, and $f_j + f_k$.

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