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A statistical uncertainty principle for estimating the time of a discrete shift in the mean of a continuous time random process

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A R T I C L E I N F O

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A B S T R A C T

The purpose of this article is to present a statistical uncertainty principle that can be used when localizing a single change in the mean of a band-limited stationary random process. The statistical model investigated is a continuous time process that experiences a shift in its mean. This continuous time process is presumed to be sampled using an ideal low-pass filter. The least squares estimate of the location of the change in mean is asymptotically Gaussian. The standard deviation of the least squares estimate of the location of the change-point provides a physical limit to the accuracy of the estimate of the time of the mean shift which cannot be bettered.

1. Introduction

Whether the world changes abruptly or gradually is a question that has absorbed scholars for centuries. In economics, structural change has been an ongoing problem in attempts to forecast key variables using econometric models. Abrupt shifts in variables are frequently overlooked or their timing misjudged, inducing large forecasting errors. Hinich et al. (2006) argue that time series data must be presumed to be generated by complex systems. The data that we generally have at our disposal is a discrete sample from a continuous time process that is subject to both ongoing and abrupt structural changes. This poses significant problems in trying to locate a point in time when an abrupt change occurs. Our goal here is to present a statistical uncertainty principle that can be used when localizing a single change in the mean of a band-limited stationary random process.

The statistical theory used to detect and estimate structural change has evolved over at least sixty years, beginning with Shewhart (1931, Ch. 19–20)) and the seminal contributions of Page (1954, 1955, and 1957), Chernoff and Zacks (1964) and Hinich and Farley (1966). These papers stimulated research on the following related, but distinct, problems.

First, estimation and inference about the change in the mean of a stationary random process was examined in Farley and Hinich (1970a, b), Hinkley (1970), Hawkins (1977), Hsu (1979), Talwar (1983), Worsley (1986), Ritov (1990), Bai (1994), Bai et al., (1998) and Lavielle and Moulines (2000). Second, estimation and inference about regime shifts associated with intersecting or "broken line" regressions were examined in Hudson (1966), Hinkley (1969, 1971) and Feder (1975a, 1975b). Third, the determination of the location of a change in the position and slope of a linear statistical model was examined in Chow (1960), Quandt (1960), Brown et al. (1975), Farley et al. (1975), Feder (1975a, 1975b), James et al. (1987), Kim and Siegmund (1989), Andrews (1993), Bai (1994, 1996), Kim (1994) and Yashchin (1995, 1997).

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Formally, we investigate the least squares estimate of the unknown time of an abrupt shift in the mean of a stationary white noise random process, when it is known that one such change has occurred in the time period over which the process is observed and sampled. If the noise is Gaussian, then the least squares estimate is maximum likelihood and this is optimal as the sample size goes to infinity (Cox and Hinkley (1974, Sect. 9.2)). Because of the ideal statistical model used, the large sample standard deviation of this maximum likelihood estimate of the time of shift serves as a lower bound to the estimate of the shift time for more complicated mean shift problems.

When addressing the three above-mentioned problems, in line with the statistical theory underpinning time series analysis more generally, the starting point is a discrete-time model. This contrasts with engineering and science applications of statistical signal processing methods, where it is understood that any discrete-time series is the result of filtering an observed signal and then decimating the filtered output (discrete-time sampling) to obtain the discrete-time sample. The nth observation $t_n=n\tau$ for the sampling frequency $f_s=1/\tau$ is either implicitly or explicitly assumed to be the true value of the process at time t_n in the standard statistical time series literature, e.g., see Grenander and Rosenblatt (1957, p.57), Box and Jenkins (1970, pp.399–400), Brillinger (1975, p. 178), Hannan (1970, Sect. II3), Fuller (1995, Sect. 1.3) and Shumway and Stoffer (2000, Sect. 1.3). This assumption is highly questionable. Any discrete-time observation is the result of some smoothing of the underlying process. The discrete-time observation is the average of a continuous time process in a time slice around t_n.

In the ideal framework employed here, the objective of the filtering operation is to remove the high-frequency components of the signal in order to remove aliasing. The rule applied is: when its frequency is higher than twice the sampling rate, a component is eliminated. If the data is sampled too slowly and is thus aliased, the high-frequency structure will be lost and, in the current context, the mean shift would be distorted and become much more difficult to localize. As such, the ideal framework developed in the article constitutes an optimal sampling experiment. In particular, we are emphasizing the direct link between the ability to accurately locate a shift in mean and knowledge and/or control over both the sampling and noise processes.[1]

With regard to the confusion about how discrete-time observations are generated, a key result is that we are able to refute a central proposition in the literature that there is no consistent estimator of change time, as argued, for example, in Hinkley (1970), Feder (1975a, b), Hawkins (1977), Worsley (1986) and Ritov (1990). Instead, we formulate the ideal problem as a continuous time process with a mean shift that is low-pass filtered and sampled at the Nyquist rate. This produces a discrete time problem where the mean shift has been smoothed by the integral of the impulse response of the filter.

If we assume that the impulse response is known, the location of the mean shift can be found and is estimable using conventional methods. This rules out the need to deal with complications arising from non-standard theory based on Brownian motion as cited, for example, in Hinkley (1970), Hawkins (1977), Kim and Siegmund (1989), Bai (1994, 1996), Kim (1994), Lavielle and Moulines (2000) and Chong (2001).

Our result will also represent a statistical version of an uncertainty principle. Given our ideal framework (incorporating both optimal filter and sampling designs), we will provide a lower bound on the estimate of the standard deviation of the location (time) of the shift in mean that cannot be improved upon (i.e. reduced). The mean shift problem considered is the simplest in the class of abrupt shifts in the parametric structure of time series. As such the bound we develop can be viewed as a first step in demonstrating that the error in detecting the time of a shift in mean is usually much larger and more complicated than is commonly thought, and is linked inextricably to consideration of the nature of sampling process that generates the time series in question. The technical details of the sampling and filtering issues are addressed in the next section.

2. Sampling a continuous time signal

We define a discrete-time sampled process $\{y(t_n)\}$ as a continuous time signal $\{x(t)\}$ that has been "discretized" by some measurement procedure. The measurement procedure we adopt is standard in signal processing. The continuous time signal $x(t)$ is filtered by a low-pass filter to remove all frequency components above a cutoff frequency f_o and then the filtered signal is sampled at a rate equal to $2f_o$ in order to avoid aliasing the sampled signal (Priestley (1981, Sect. 7.1.1)).

Suppose that the filter is *linear*. Then the filter is characterized by its *impulse response* function denoted $h(t)$. The output of the filter can be represented as

$$y(t_n)=\int_{-\infty}^{\infty} h(s)x(t-s)ds \tag{1}$$

The filter smoothes the input process $x(t)$ by removing all frequency components of the input that exceed the cutoff frequency f_o. In this article, we let the impulse response have unit area to simplify notation.

[1] Indeed, it is our contention that the stated ability of many studies to accurately locate a mean shift is in fact largely illusory because of the lack of knowledge and control over the sampling and noise processes—facts that have not been typically acknowledged in the broader literature.

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The framework introduced above rests on the assertion that the defining characteristic of the underlying process is a continuous time process—a notion that can be readily extended to economic systems. In the context of economic systems, for example, the continuity of the underlying process is related to the notion of continuity of exchange.

In economic applications, macroeconomists use data that is typically supplied by national statistical agencies.[2] In general, this data is compiled from surveys that are often conducted over a week's duration either once a month or once a quarter or perhaps from (aggregated) taxation data submitted on a quarterly basis. The continuity of economic activity associated with the global economy that underpins this reporting activity by economic agents (included in the surveys) reflects the normal day to day activities of businesses and other economic agents.

The results of these surveys contain information that is benchmarked with additional information obtained from annual surveys (or censuses) which are then used to generate both quarterly and annual national accounts statistics. This process can be viewed as a very complicated, albeit imperfect, filtering process that works to smooth out any abrupt changes in the economy.

For example, any abrupt change appearing within a quarter will show up as a change across consecutive quarters. But the information content in the resultant quarterly time series data will not be sufficient to locate and model the change as it appeared within the quarter in which it occurred. Therefore, technical issues concerning sampling and filtering have to be addressed when trying to locate when structural change has occurred in time series data. In general, some sort of filtering operation is always latent in social science applications even if the investigator believes that each observation at t_n is the true value of the process at that time. The fundamental uncertainty principle in the natural world also applies for all measurements in the social sciences. In particular, it should be recognized that the consequences of inherent limitations to coding, transmitting and analyzing information on institutional and other forms of organizational behavior constitutes an important source of finite bandwidth in social systems.

One implication of this finite bandwidth property is its apparent inconsistency with the conventional requirement of infinite bandwidth $(-\infty, \infty)$ underpinning the conception of continuous white noise, typically employed in continuous time econometric and time series problems (see, for example, Bergstrom, (1976) and Priestley, (1981, pp. 234–235)). Note further that the assumption of infinite bandwidth also ensures that the sampling interval $\tau = (2f_o)^{-1} \to 0$ as $f_o \to \infty$. Since all real signals are band-limited, the appropriate continuous white noise concept is band-limited white noise. The spectral density of a band-limited white noise signal is constant over the finite-pass band range $(-f_o, f_o)$ and zero outside this range.

A key consequence of the finite bandwidth property is that it is impossible to obtain a precise measurement of a process at a precise point in time—we cannot observe or force (collapse) $\tau \to 0$ because the finite bandwidth property strictly ensures that $f_o < \infty$, by definition.

In this article we do not address the inherent error in the time of measurement. We treat t_n as the true time of measurement but observed time series data $x(t_n)$ is really $y(t_n)$ for some filtering operation with a usually unknown impulse response. For the theoretical development of our ideas we adopt the ideal situation and assume that the impulse response is *known* in order to find the optimal least squares estimate of the time of the mean shift (i.e. shift time). Any error in the impulse response function used to estimate the shift time or inherent error in the time of measurement increases the error of the estimate of the shift time.

In order to establish the minimum mean squared error of any estimate of the shift time, there must be a precise statement of the statistical problem. This is done in the following section.

3. The least squares estimate of the shift time

The idealized statistical problem is as follows. Suppose that a continuous time signal $x(t)$ has an abrupt shift in mean from μ to $\mu+\delta$ at an *unknown* time ξ. The continuous time process with mean shift can be represented as

$$x(t) = \delta I_{[\xi,\infty]}(t) + u(t) \tag{2}$$

where $u(t)$ is an underlying continuous time white noise process, δ denotes the shift in mean and $I_{[\xi,\ \infty]}$ is an "Indicator" function. Given (1) and (2), the sampled output obtained after using the ideal filtering operation on (2), denoted $\{y(t_n)\}$, can be represented as

$$\begin{aligned} y(t_n) &= \int_{-\infty}^{\infty} h(s)\{\delta I_{[\xi,\infty]}(t-s) + u(t-s)\}ds \\ &= \delta \int_{-\infty}^{\infty} h(s) I_{[\xi,\infty]}(t-s)ds + \int_{-\infty}^{\infty} h(s)u(t-s)ds \\ &= \delta \int_{-\infty}^{t-\xi} h(s)ds + e(t_n) \end{aligned} \tag{3}$$

It is apparent from inspection of (3) that the sampled process can be viewed as comprising two components. The first component is the shift in the mean of the filtered process given by the term $\delta\int_{-\infty}^{t-\xi} h(s)ds$. This term captures the effect of the

[2] This data is typically aliased because it is under sampled.

mean shift δ being smoothed by the integral of the impulse response of the filter. The second component $e(t_n)$ is band-limited white noise obtained from filtering the continuous white noise signal $u(t)$ in (2) at cutoff frequency f_o.

The filtering operation in (3) can be interpreted as filtering the continuous time signal in (2) by a constant gain low-pass filter whose complex frequency response is $H(f)=1$ in the band $(-f_o, f_o)$ and zero outside the band. The impulse response of this ideal constant gain filter is the sinc function $(\pi t)^{-1}\sin(\pi\tau^{-1}t)$ where $\tau=(2f_o)^{-1}$. This filtered signal is then sampled at the rate $2f_o$ producing a set of observations at the discrete times t_n such that $\{y(t_n)\}=\{y(t_1), y(t_2), \ldots, y(t_{N-1}), y(t_N)\}$ where $t_n=n\tau$ and $T=N\tau$ for an even integer N. Thus, the sampled process $\{y(t_n)\}$ can be interpreted as band-limited white noise with a shift in the mean δ at an unknown time ξ.

Since the filtered stochastic component is band-limited white noise, the sampled noise process $\{e(t_n)\}$ is a sequence of uncorrelated random variables for $n=1, \ldots, N$. If $e(t_n)$ is Gaussian then the $y(t_n)$ are independently distributed Gaussian variates with common variance denoted by σ_e^2.[3]

The sample size N is directly linked to the bandwidth frequency f_o for a fixed value of T. For example, the larger the frequency f_o (implying cycles of short duration), the smaller will be the sampling interval τ and the larger will be the required sample size N for a given value of T. This implies that the higher the frequency of the elements, the higher must be the rate of sampling for a given value of T. However for given f_o and τ, if we want to observe cycles of long duration then N must increase.

The shift in the mean of the sampled output is $\delta\int_{-\infty}^{n\tau-\xi}(\pi s)^{-1}\sin(\pi\tau^{-1}s)ds$. Since $\int_{-\infty}^{n\tau}(\pi s)^{-1}\sin(\pi\tau^{-1}s)ds=\int_{-\infty}^{n}\nu^{-1}\sin(\pi\nu)d\nu$ the shift in the mean is $\delta F(n-\alpha)$ where $\alpha=\tau^{-1}\xi$ is the shift time normalized by the sampling unit and

$$F(n-\alpha)=\int_{-\infty}^{n-\alpha}\frac{\sin(\pi\nu)}{\pi\nu}d\nu \tag{4}$$

We derive the least squares estimate of α from the sample $\{y(t_1), \ldots, y(t_N)\}$ where $y(t_n)=F(\eta-\alpha)+e(t_n)$ assuming that the underlying noise is Gaussian. It then follows that the sampled noise values $e(t_n)$ are Gaussian, independently distributed variates with common variance σ_e^2.

Recall that the goal of this enterprise is to find the lowest possible variance for an estimate of the shift time ξ using ideal assumptions. The highest probability of detection of the shift is when the shift occurs in the middle of the sampling period since there are an equal number of observations of the process before and after the shift. Setting the shift time at the middle also reduces the error of the estimate since detecting a shift and localizing its time are highly interrelated.

The derivative of $\sum_{n=1}^{N}F^2(n-\alpha)$ with respect to α is constant at the shift and so the least squares estimate $\hat{\alpha}$ of α is the value that maximizes the statistic

$$S(\alpha)=\sum_{n=1}^{N}F(n-\alpha)y(t_n) \tag{5}$$

If the noise is Gaussian then $\hat{\xi}=\tau\hat{\alpha}$ is the maximum likelihood estimate of ξ and is thus efficient (Rao (1965, pp. 289–290)). The large sample variance of $\hat{\xi}$ is

$$Var(\hat{\xi})=\frac{\sigma_e^2}{\delta^2\sum_{n=1}^{N}\left[\frac{dF(n-\tau^{-1}\xi)}{d\xi}\right]^2}=\frac{\sigma_e^2}{\frac{\delta^2}{\tau^2}\sum_{n=1}^{N}\frac{\sin^2[\pi(n-\tau^{-1}\xi)]}{[\pi(n-\tau^{-1}\xi)]^2}} \tag{6}$$

Since ξ is in the middle of the observation period, $\sum_{n=1}^{N}\frac{\sin^2[\pi(n-\tau^{-1}\xi)]}{[\pi(n-\tau^{-1}\xi)]^2}=1$ and thus from (6), $Var(\hat{\xi})=\left(\frac{\sigma_e}{2\delta f_o}\right)^2$.

Therefore the optimal standard deviation of the estimate is

$$\sigma(\hat{\xi})=\frac{\sigma_e}{2\delta f_o}. \tag{7}$$

This simple result can be generalized to any non-Gaussian white noise process where the sampled noise values $e(t_n)$ are independently distributed, rather than being merely uncorrelated. If σ_e^2 is finite then from Theorem 27.1 in Billingsley (1979), the statistic $S(\alpha)$ is asymptotically Gaussian as $N\to\infty$. This result is important because it permits statistical inference using conventional distribution theory instead of the non-standard theory based upon Brownian motion. Within the accuracy of this standard approximation of a statistic the standard deviation of the least squares estimate $\hat{\xi}$ is given by (7).

4. Conclusions

In this article, we have presented a statistical uncertainty principle associated with localizing a single change in the mean of a band-limited stationary random process. The statistical model used was a continuous time process that was assumed to experience a shift in its mean. An ideal low-pass filter was used to derive discrete-time observations from the

[3] The variance of the sampled process in (3) can be demonstrated to equal $2f_0\sigma_u^2$ where σ_u^2 is the variance of the continuous time white noise process in (2).

continuous time process producing a "discretized" band-limited white noise process with a shift in mean at an unknown time that was to be estimated.

In developing our approach, we emphasized the direct link between the ability to accurately locate a shift in mean and knowledge and/or control over both the sampling and noise processes.

A key result was that the least squares estimate of the location of the change in mean was demonstrated to be asymptotically Gaussian. Given the ideal assumptions made about the sampling process, noise properties and location of the mean shift (to ensure the highest probability of detection), the optimal approximation (7) of the standard deviation of the least squares estimate of the location of the change-point provided a physical limit to the accuracy of the estimate of the time of the mean shift which could not be bettered. In particular, if the sampling system response was not the ideal low-pass filter used to derive the result, then the noise would not be white and the accuracy of the least squares estimate would be worse than that in the ideal case.

A simple example is helpful to understand an implication of this result. If τ is a day and $\delta=\sigma_e/2$, the ideal two standard deviation uncertainty of the estimate of the mean shift will be about four days.

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References

Andrews, D.W.K., 1993. Tests for parameter instability and structural change with unknown change point. Econometrica 61, 821–856.
Bai, J., 1994. Least squares estimation of a shift in linear processes. J. Time Ser. Anal. 15, 453–472.
Bai, J., 1996. Testing for parameter constancy in linear regressions: an empirical distribution function approach. Econometrica 64, 579–622.
Bai, J., Lumsdaine, R.L., Stock, J.H., 1998. Testing for and dating common breaks in multivariate time series. Rev. Econ. Stud. 65, 395–432.
Bergstrom, A.R. (Ed.), 1976. Statistical Inference in Continuous Time Economic Models. North-Holland, Amsterdam.
Billingsley, P., 1979. Probability and Measure. John Wiley, New York.
Box, G.E.P., Jenkins, G.M., 1970. Time Series Analysis: Forecasting and Control. Holden-Day, San Francisco.
Brillinger, D., 1975. Time Series, Data Analysis and Theory. Holt, Rinehart, and Winston, New York.
Brown, R.L., Durbin, J., Evans, J.M., 1975. Techniques for testing the constancy of regression relations over time. J. Roy. Stat. Soc. B. 37, 149–192.
Chernoff, H., Zacks, S., 1964. Estimating the current mean of a normal distribution which is subjected to changes in time. Ann. Math. Stat. 35, 999–1018.
Chong, T.T.L., 2001. Structural change in AR(1) models. Econometric Theory 17, 87–155.
Chow, G.C., 1960. Tests of equality between sets of coefficients in two linear regressions. Econometrica 28, 591–605.
Cox, D.R., Hinkley, D.V., 1974. Theoretical Statistics. Chapman and Hall, London.
Farley, J.U., Hinich, M.J., 1970a. A test for a shifting slope coefficient in a linear model. J. Amer. Statistical Assoc. 65, 1320–1329.
Farley, J.U., Hinich, M.J., 1970b. Detecting "small" mean shifts in time series. Manage. Sci. 17, 189–199.
Farley, J.U., Hinich, M.J., McGuire, T.W., 1975. Some comparisons of tests for a shift in the slopes of a multivariate linear time series model. J. Econometrics 3, 297–318.
Feder, P.I., 1975a. On asymptotic distribution theory in segmented regression problems-identified case. Ann. Stat. 3, 49–83.
Feder, P.I., 1975b. The log likelihood ratio in segmented regression. Ann. Stat. 3, 84–97.
Fuller, W.A., 1995. Introduction to Statistical Time Series, 2nd ed. Wiley, New York.
Grenander, U., Rosenblatt, M., 1957. Statistical Analysis of Stationary Time Series. Wiley, New York.
Hannan, E.J., 1970. Multiple Time Series. Wiley, New York.
Hawkins, D.M., 1977. Testing a sequence of observations for a shift in location. J. Amer. Statistical Assoc. 72, 180–186.
Hinich, M.J., Farley, J.U., 1966. Theory and application of an estimation model for time series with nonstationary means. Manage. Sci. 12, 648–658.
Hinich, M.J., Foster, J., Wild, P., 2006. Structural change in macroeconomic time series: a complex systems perspective. J. Macroecon. 28.
Hinkley, D.V., 1969. Inference about the intersection in two-phase regression. Biometrika 56, 495–504.
Hinkley, D.V., 1970. Inference about the change-point in a sequence of random variables. Biometrika 57, 1–17.
Hinkley, D.V., 1971. Inference in two-phase regression. J. Amer. Statistical Assoc. 66, 736–743.
Hsu, D.A., 1979. Detecting shifts of parameter in gamma sequences with application to stock price and air flow analysis. J. Amer. Statistical Assoc. 74, 31–40.
Hudson, D.J., 1966. Fitting segmented curves whose join points have to be estimated. J. Amer. Statistical Assoc. 61, 1097–1129.
James, B., James, K.L., Siegmund, D., 1987. Tests for a change-point. Biometrika 74, 71–84.
Kim, H.J., Siegmund, D., 1989. The likelihood ratio test for a change-point in simple linear regression. Biometrika 76, 409–423.
Kim, H.J., 1994. Likelihood ratio and cumulative sum tests for a change-point in linear regression. J. Multivariate. Anal. 51, 54–70.
Lavielle, M., Moulines, E., 2000. Least-squares estimation of an unknown number of shifts in a time series. J. Time. Ser. Anal. 21, 33–59.
Page, E.S., 1954. Continuous inspection schemes. Biometrika 41, 100–114.
Page, E.S., 1955. A test for a change in a parameter occurring at an unknown point. Biometrika 42, 523–527.
Page, E.S., 1957. On problems in which a change in parameter occurs at an unknown point. Biometrika 44, 248–252.
Priestley, M.B., 1981. Spectral Analysis and Time Series. Academic Press, London.
Quandt, R.E., 1960. Tests of the hypothesis that a linear regression obeys two separate regimes. J. Amer. Statistical Assoc. 55, 324–330.
Rao, C.R., 1965. Linear Statistical Inference and Its Applications. Wiley, New York.
Ritov, Y., 1990. Asymptotic efficient estimation of the change point with unknown distributions. Ann. Stat. 18, 1829–1839.
Shewhart, W.A., 1931. Economic Control of Quality of Manufactured Product. Van Nostrand, New York.
Shumway, R.H., Stoffer, D.S., 2000. Time Series Analysis and Its Applications. Springer, New York.
Talwar, P.P., 1983. Detecting a shift in location. J. Econometrics 23, 53–367.
Worsley, K.J., 1986. Confidence regions and tests for a change-point in a sequence of exponential family random variables. Biometrika 73, 91–104.
Yashchin, E., 1995. Estimating the current mean of a process subject to abrupt changes. Technometrics 37, 311–323.
Yashchin, E., 1997. Change-point models in industrial applications. Nonlinear Anal.: Theory 7, 3997–4006.

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