An omnibus test for time series model I(d)

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February 2, 2008

Abstract

The fractional integrated process has been widely studied since the seminal work of Granger and Joyeus (1980). However, most of the tests for fractional integration are time-domain tests. This paper contributes to the

*We would like to thank Jianqing Fan, N.H. Chan and Yin-Wong Cheung for helpful comments. We also thank Carrella Ernesto and Lumpkin Samuel Mcspadden for able research assistance. Corresponding author: Terence Chong, Department of Economics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong. Email: chong2064@cuhk.edu.hk. Webpage: http://www.cuhk.edu.hk/eco/staff/tlchong/tlchong3.htm.

literature by developing a Cramér-von Mises type frequency-domain test for fractional integration. Our test is shown to have high power against a wide variety of alternatives.

Key words: Spectral distribution; Cramér-von Mises test; Long memory; Fractionally integrated process.

1. Introduction

The fractional integration process has found wide applications in various academic disciplines. A number of economic and financial time series have been found to be fractionally integrated. For example, the consumption time series (Haubrich, 1993), asset prices (Ding et al., 1993), stock returns (Lo, 1991), exchange rates (Diebold et al.,1991), interest rates (Backus and Zin, 1993) and inflation rates (Baillie et al., 1996). Despite it extensive applications, rigorous tests for whether a time series falls into the class of fractional integration are surprisingly under developed. Recently, Hinich and Chong (2007) have developed a time-domain class test which can distinguish fractionally integrated processes from other time series processes. In this paper, we extend the work of Hinich and Chong (2007) to the frequency domain. A Cramér-von Mises type test is developed.

To begin with, consider a covariance stationary time series process $\{y_t\}$, let

$$\sigma(h) = E(y_t - \mu)(y_{t+h} - \mu), \qquad h = 0, \pm 1, \pm 2, ... \tag{1.1}$$

$$\rho_h = \frac{\sigma(h)}{\sigma(0)}. \tag{1.2}$$

The standardized spectral density is defined as

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \rho_h \cos \lambda h \qquad -\pi < \lambda \leq \pi. \tag{1.3}$$

Since $f(\lambda) = f(-\lambda)$, the standardized spectral distribution can be written as

$$F(\lambda) = 2 \int_0^{\lambda} f(\upsilon)\, d\upsilon = \frac{1}{\pi} \left(\lambda + 2 \sum_{h=1}^{\infty} \rho_h \frac{\sin \lambda h}{h} \right), \tag{1.4}$$

with $F(\pi) = 1$.

The standardized spectral distribution has the properties of a probability distribution on $[0, \pi]$. The sample autocovariance is

$$c_h = c_{-h} = \frac{1}{T} \sum_{t=1}^{T-h} (y_t - \mu)(y_{t+h} - \mu), \tag{1.5}$$

where μ is the true mean of the process. Let the sample autocorrelation sequence be

$$r_h = \frac{c_h}{c_0}. \tag{1.6}$$

The standardized sample spectral density is

$$I_T(\lambda) = \frac{1}{2\pi} \sum_{h=-(T-1)}^{T-1} r_h \cos \lambda h, \qquad -\pi < \lambda \leq \pi, \tag{1.7}$$

and the standardized sample spectral distribution function is

$$F_T(\lambda) = 2 \int_0^{\lambda} I_T(\upsilon)\, d\upsilon = \frac{1}{\pi} \left(\lambda + 2 \sum_{h=1}^{T-1} r_h \frac{\sin \lambda h}{h} \right). \tag{1.8}$$

Similar to Anderson et al. (2004), we construct a test statistic as follows:

$$\begin{aligned} Z_T(\lambda) &= \sqrt{T}\left[F_T(\lambda) - F(\lambda)\right] \\ &= \frac{2\sqrt{T}}{\pi}\left(\sum_{h=1}^{T-1} r_h \frac{\sin \lambda h}{h} - \sum_{h=1}^{\infty} \rho_h \frac{\sin \lambda h}{h}\right) \\ &= \frac{2\sqrt{T}}{\pi}\left(\sum_{h=1}^{T-1} (r_h - \rho_h) \frac{\sin \lambda h}{h} - \sum_{h=T}^{\infty} \rho_h \frac{\sin \lambda h}{h}\right). \end{aligned} \tag{1.9}$$

$Z_T(\lambda)$ can be rewritten as

$$Y_T(u) = \sqrt{T}\left[F_T(\lambda(u)) - F(\lambda(u))\right], \tag{1.10}$$

where

$$\lambda(u) = G^{-1}\left[G(\pi)\, u\right], \tag{1.11}$$

$$G(\lambda) = 2\int_0^{\lambda} f^2(v)\, dv \tag{1.12}$$

and

$$u = \frac{G(\lambda)}{G(\pi)}. \tag{1.13}$$

To convert the original $Z_T(\lambda)$ process into a process on the zero-one interval, let

$$Y_T^*(u) = \frac{Y_T(u)}{2\sqrt{\pi G(\pi)}}. \tag{1.14}$$

Define the Cramér-von Mises statistic as

$$W_T^2 = \int_0^1 \left[Y_T^*(u)\right]^2 du. \tag{1.15}$$

Let

$$W_T^2 = \frac{T}{4\pi^4 G^2(\pi)} \sum_{h=1}^{H} \left(\sum_{g=1}^{T-1} \frac{(r_g - \rho_g)(\rho_{h+g} - \rho_{h-g})}{g} \right)^2. \tag{1.16}$$

H is selected so that the summation term is negligible for $h > H$. The asymptotic distribution of W_T^2 can be found in Anderson and Darling (1952) and Anderson et al. (2004).

2. Testing for Fractional Integration

This paper contributes to the literature by deriving a frequency-domain Cramér-von Mises test for fractional integration. For ease of illustration, we discuss the case of a simple $I(d)$ process:

$$(1-L)^d (y_t - \mu) = u_t, \qquad (t = 1, 2, ..., T), \tag{2.1}$$

where $\{u_t\} \sim i.i.d.(0, \sigma^2)$. The fractional difference operator $(1-L)^d$ is defined by its Maclaurin series

$$(1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)} L^j, \tag{2.2}$$

where L is the lag operator, $\Gamma(x)$ is the Euler gamma function defined as

$$\Gamma(x) = \int_0^{\infty} z^{x-1} \exp(-z)\, dz \qquad \text{for } x > 0,$$

$$\Gamma(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(x+k)\,k!} + \int_1^{\infty} z^{x-1} \exp(-z)\, dz \qquad \text{for } x < 0, x \neq -1, -2, -3, ...$$

The process has long memory when d is positive. There has been a great stride forward in the estimation of the long memory model in recent years (Chong, 2006; Phillips and Shimotsu, 2005). Different tests for long memory in the time domain have been developed (Hinich and Chong, 2007; Chen and Deo, 2004; Tanaka, 1999). For a comprehensive review of the literature, one is referred to Beran (1994), Baillie (1996) and Henry and Zaffaroni (2002). In our case, the process can be rewritten as

$$y_t - \mu = \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(d)\,\Gamma(j+1)} u_{t-j}, \qquad (t = 1, 2, ..., T). \tag{2.3}$$

The process is stationary if $d < 0.5$. For $-0.5 < d < 0.5$, Hosking (1996) shows that

$$T^{0.5-d}(\overline{y} - \mu) \xrightarrow{d} N\left(0, \frac{\sigma^2 \Gamma(1-2d)}{(1+2d)\,\Gamma(1+d)\,\Gamma(1-d)}\right), \tag{2.4}$$

where $\overline{y}$ is the sample mean and the j^{th} autocorrelation of this $I(d)$ process is given by

$$\rho_j(d) = \prod_{i=1}^{j} \frac{d+i-1}{i-d} \qquad (j = 1, 2, ...). \tag{2.5}$$

Given the value of the differencing parameter, the standardized spectral density is

$$f(\lambda|d) = \frac{1}{2\pi}\left|2\sin\left(\frac{\lambda}{2}\right)\right|^{-2d}, \qquad -\pi \le \lambda \le \pi \tag{2.6}$$

and the standardized spectral distribution is

$$F(\lambda|d) = 2\int_0^{\lambda} \frac{1}{2\pi}\left(2\sin\left(\frac{\upsilon}{2}\right)\right)^{-2d} d\upsilon, \qquad 0 \le \lambda \le \pi. \tag{2.7}$$

We test

$$H_0 : y_t \text{ is an } I(d)$$

against

$$H_1 : y_t \text{ does not follow an } I(d)$$

From (1.12), we have

$$G(\lambda|d) = 2\left(\frac{1}{2\pi}\right)^2 \int_0^{\lambda} \left(2\sin\left(\frac{\upsilon}{2}\right)\right)^{-4d} d\upsilon, \qquad 0 \le \lambda \le \pi. \tag{2.8}$$

$$G(\pi|d) = \frac{B\left(\frac{1}{2}, \frac{1}{2} - 2d\right)}{2^{1+4d}\pi^2}, \tag{2.9}$$

where $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function. Thus,

$$
\begin{aligned}
W_T^2(d) &= \frac{T}{4\pi^4 G^2(\pi|d)} \sum_{h=1}^{H} \left(\sum_{g=1}^{T-1} \frac{(r_g - \rho_g(d))(\rho_{h+g}(d) - \rho_{h-g}(d))}{g} \right)^2 \\
&= 2^{8d} T B^{-2}\left(\frac{1}{2}, \frac{1}{2} - 2d\right) \times \\
&\sum_{h=1}^{H} \left(\sum_{g=1}^{T-1} g^{-1} \left(r_g - \prod_{i=1}^{h+g} \frac{d+i-1}{i-d} \right) \left(\prod_{i=1}^{h+g} \frac{d+i-1}{i-d} - \rho_{h-g}(d) \right) \right)^2,
\end{aligned}
\tag{2.10}
$$

where

$$
\begin{aligned}
\rho_{h-g}(d) &= \prod_{i=1}^{|h-g|} \frac{d+i-1}{i-d} \qquad \text{if } h \neq g; \\
&= 1 \qquad \text{if } h = g.
\end{aligned}
$$

When $d = 0$, the test can be simplified to

$$
W_T^2(0) = \frac{T}{\pi^2} \sum_{h=1}^{T-1} \frac{r_h^2}{h^2}, \tag{2.11}
$$

where $\sqrt{T} r_h$ has a limiting standard normal distribution. The limiting distribution of $W_T^2(0)$ is

$$
\sum_{i=1}^{\infty} \frac{1}{(\pi i)^2} X_i^2, \tag{2.12}
$$

where X_i are independent standard normal random variables. (2.12) is the limiting distribution of the Cramér-von Mises statistic (Stephens, 1986).

When d is known, we can perform the test via the following steps:

1. Calculate the r_g for $g = 1, 2, ..., T-1$.
2. Calculate $W_T^2(d)$.
3. The null of fractional integration will be rejected at α if $W_T^2(d)$ exceeds the critical value in Table 1 in the next section.

If the value of d is unknown, we can replace it by a consistent estimate. The estimators can be parametric (Sowell, 1992; Tieslau et al., 1996; Chong, 2000) or semiparametric (Velasco, 1999; Phillips and Shimotsu, 2005). The Cramér-von Mises criterion is

$$
\begin{aligned}
& \frac{T}{2\pi^2 G^2\left(\pi|\widehat{d}\right)} \int_0^{\pi} \left(F_T(\lambda) - F\left(\lambda|\widehat{d}\right)\right)^2 f^2\left(\lambda|\widehat{d}\right) d\lambda \\
\sim\; & \frac{T}{4\pi^4 G^2\left(\pi|\widehat{d}\right)} \sum_{h=1}^{\infty} \left(\sum_{g=1}^{T-1} \frac{\left(r_g - \rho_g\left(\widehat{d}\right)\right)\left(\rho_{h+g}\left(\widehat{d}\right) - \rho_{|h-g|}\left(\widehat{d}\right)\right)}{g} \right)^2 .
\end{aligned}
$$

Thus, we define

$$
\begin{aligned}
W_T^2\left(\widehat{d}\right) &= \frac{T}{4\pi^4 G^2\left(\pi|\widehat{d}\right)} \sum_{h=1}^{H} \left(\sum_{g=1}^{T-1} \frac{\left(r_g - \rho_g\left(\widehat{d}\right)\right)\left(\rho_{h+g}\left(\widehat{d}\right) - \rho_{h-g}\left(\widehat{d}\right)\right)}{g} \right)^2 \\
&= 2^{8\widehat{d}} T B^{-2}\left(\frac{1}{2}, \frac{1}{2} - 2d\right) \times \\
& \sum_{h=1}^{H} \left(\sum_{g=1}^{T-1} g^{-1} \left(r_g - \prod_{i=1}^{h+g} \frac{\widehat{d} + i - 1}{i - \widehat{d}} \right) \left(\prod_{i=1}^{h+g} \frac{\widehat{d} + i - 1}{i - \widehat{d}} - \rho_{h-g}\left(\widehat{d}\right) \right) \right)^2 ,
\end{aligned}
\tag{2.13}
$$

where H is selected so that the term in the sum is negligible for $h > H$, and

$$\begin{aligned}\rho_{h-g}\left(\widehat{d}\right) &= \prod_{i=1}^{|h-g|} \frac{\widehat{d}+i-1}{i-\widehat{d}} \qquad \text{if } h \neq g; \\ &= 1 \qquad \text{if } h = g.\end{aligned}$$

3. Simulations

Experiment 1. This experiment generates the critical values of the test when d is known. Consider the following model:

$$(1-L)^d (y_t - \mu) = u_t, \qquad t = 1, 2, ..., T.$$

$$u_t \sim N(0, 1).$$

Samples of size $T = 20, 50$, 100, 200 and 2000 are simulated to study the finite-sample behavior of the test. We set $H = 10000$ and $d = -0.4, -0.3, -0.2, -0.1, 0, 0.1$ and 0.2. Without loss of generality, we assume $\mu = 0$. For each value of T, d, we simulate the test statistic with 200000 replications. For known and unknown μ, Tables 1a and 1b report the critical value c of the finite-sample distribution of $W_T^2(d)$ such that $\Pr(W_T^2(d) \geq c) = \alpha$, for $\alpha = 0.5, 0.25, 0.1, 0.05, 0.025, 0.01, 0.005$ and 0.001.

Table 1a: Critical values of the test with known μ.

d	$T\backslash\alpha$	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.001
-0.4	20	0.043	0.084	0.145	0.191	0.239	0.301	0.349	0.447
	50	0.050	0.095	0.162	0.218	0.274	0.348	0.405	0.540
	100	0.052	0.099	0.170	0.228	0.289	0.372	0.432	0.584
	200	0.053	0.101	0.174	0.232	0.294	0.381	0.449	0.596
	2000	0.054	0.102	0.177	0.239	0.302	0.390	0.458	0.621
-0.3	20	0.057	0.109	0.188	0.251	0.314	0.392	0.450	0.586
	50	0.065	0.123	0.210	0.282	0.355	0.456	0.534	0.704
	100	0.068	0.128	0.219	0.292	0.370	0.477	0.559	0.749
	200	0.069	0.130	0.224	0.300	0.379	0.487	0.572	0.771
	2000	0.071	0.132	0.225	0.303	0.385	0.497	0.581	0.791
-0.2	20	0.071	0.136	0.233	0.311	0.391	0.494	0.568	0.738
	50	0.080	0.152	0.259	0.347	0.439	0.564	0.652	0.875
	100	0.084	0.157	0.268	0.360	0.453	0.585	0.683	0.916
	200	0.086	0.160	0.274	0.367	0.465	0.596	0.696	0.921
	2000	0.088	0.162	0.276	0.370	0.470	0.604	0.709	0.933
-0.1	20	0.083	0.158	0.270	0.360	0.453	0.572	0.662	0.857
	50	0.094	0.175	0.299	0.400	0.504	0.644	0.752	1.005
	100	0.098	0.182	0.309	0.414	0.524	0.674	0.782	1.052
	200	0.100	0.185	0.314	0.418	0.528	0.683	0.798	1.079
	2000	0.102	0.188	0.319	0.424	0.539	0.689	0.808	1.093

Table 1a (continued)

d	$T\backslash\alpha$	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.001
0.0	20	0.088	0.166	0.284	0.380	0.477	0.610	0.702	0.916
	50	0.099	0.184	0.312	0.419	0.532	0.679	0.796	1.055
	100	0.104	0.192	0.325	0.433	0.548	0.703	0.817	1.089
	200	0.106	0.195	0.332	0.446	0.562	0.718	0.841	1.134
	2000	0.108	0.198	0.336	0.449	0.565	0.721	0.839	1.131
0.1	20	0.076	0.144	0.246	0.328	0.415	0.531	0.621	0.825
	50	0.088	0.165	0.282	0.380	0.485	0.628	0.746	1.007
	100	0.094	0.175	0.301	0.404	0.513	0.666	0.785	1.073
	200	0.097	0.181	0.311	0.419	0.531	0.687	0.811	1.090
	2000	0.100	0.187	0320	0.430	0.546	0.695	0.813	1.108
0.2	20	0.029	0.054	0.091	0.119	0.149	0.193	0.228	0.310
	50	0.036	0.072	0.123	0.167	0.220	0.303	0.376	0.568
	100	0.040	0.082	0.147	0.203	0.271	0.382	0.480	0.731
	200	0.043	0.090	0.165	0.230	0.307	0.436	0.560	0.884
	2000	0.050	0.111	0.213	0.302	0.402	0.563	0.708	1.077

Table 1b: Critical values of the test with unknown μ and known d.

d	$T \backslash \alpha$	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.001
-0.4	20	0.043	0.084	0.145	0.193	0.242	0.305	0.354	0.464
	50	0.050	0.095	0.163	0.218	0.275	0.351	0.410	0.552
	100	0.052	0.099	0.170	0.228	0.289	0.374	0.441	0.592
	200	0.053	0.100	0.173	0.233	0.296	0.382	0.449	0.609
	2000	0.054	0.102	0.176	0.237	0.303	0.389	0.454	0.618
-0.3	20	0.057	0.111	0.189	0.253	0.316	0.403	0.467	0.610
	50	0.065	0.123	0.211	0.283	0.357	0.456	0.533	0.710
	100	0.068	0.128	0.219	0.296	0.373	0.480	0.560	0.746
	200	0.069	0.130	0.223	0.300	0.380	0.487	0.566	0.758
	2000	0.071	0.133	0.227	0.303	0.384	0.492	0.580	0.788
-0.2	20	0.072	0.137	0.235	0.314	0.394	0.498	0.579	0.756
	50	0.081	0.152	0.258	0.345	0.435	0.562	0.656	0.885
	100	0.084	0.158	0.270	0.360	0.454	0.584	0.680	0.904
	200	0.086	0.160	0.274	0.367	0.465	0.598	0.699	0.938
	2000	0.088	0.163	0.277	0.371	0.468	0.604	0.710	0.957
-0.1	20	0.085	0.162	0.275	0.368	0.462	0.594	0.686	0.896
	50	0.095	0.176	0.301	0.403	0.508	0.651	0.762	1.011
	100	0.098	0.182	0.309	0.414	0.523	0.672	0.783	1.033
	200	0.100	0.185	0.314	0.423	0.538	0.682	0.796	1.077
	2000	0.102	0.187	0.317	0.425	0.535	0.679	0.798	1.081

Table 1b (continued)

d	$T\backslash\alpha$	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.001
0.0	20	0.094	0.174	0.295	0.391	0.491	0.622	0.718	0.939
	50	0.102	0.189	0.320	0.427	0.537	0.688	0.803	1.051
	100	0.106	0.195	0.330	0.438	0.554	0.713	0.839	1.131
	200	0.107	0.197	0.334	0.447	0.561	0.723	0.842	1.130
	2000	0.108	0.199	0.336	0.449	0.567	0.724	0.844	1.134
0.1	20	0.090	0.162	0.266	0.346	0.427	0.527	0.598	0.769
	50	0.096	0.175	0.291	0.384	0.478	0.605	0.702	0.929
	100	0.098	0.180	0.303	0.401	0.504	0.639	0.747	0.985
	200	0.099	0.183	0.310	0.411	0.518	0.662	0.772	1.041
	2000	0.099	0.185	0.316	0.424	0.534	0.686	0.811	1.072
0.2	20	0.045	0.074	0.108	0.132	0.155	0.182	0.201	0.242
	50	0.050	0.087	0.134	0.167	0.198	0.237	0.264	0.324
	100	0.049	0.094	0.151	0.192	0.231	0.281	0.318	0.398
	200	0.048	0.098	0.164	0.212	0.260	0.321	0.369	0.479
	2000	0.048	0.103	0.189	0.259	0.330	0.431	0.505	0.689

Experiment 2. When d is unknown, the following procedure is used to simulate the test statistic. We set $H = 10000$. For each fixed T ($T = 50, 100, 200, 500$ and 2000), one million values of d were drawn from the uniform distribution $U(-0.5, 0.25)$. An $I(d)$ process was generated for each value of d and T and the values of $\widehat{d}$ and $W_T^2\left(\widehat{d}\right)$ were recorded. The first-order estimator of Chong (2000) for d is used and the values of $\widehat{d}$ are binned; for instance, for $\widehat{d} = 0.1$, the values of $\widehat{d}$ between 0.05 and 0.15 were placed in a vector. The corresponding values of $W_T^2\left(\widehat{d}\right)$ were placed in another vector and percentage points were calculated. These are the points recorded against $\widehat{d} = 0.1$. The points given for other values of $\widehat{d}$ were found similarly. Only those $\widehat{d}$ whose values are between -0.45 and 0.25 were used for binning. The number of observations used for a given $\widehat{d}$ will be approximately equal to 1000000/(number of bins). Tables 2a and 2b report the critical value c such that

$$\Pr\left(W_T^2\left(\widehat{d}\right) \geq c\right) \simeq \alpha$$

for $\alpha = 0.5, 0.25, 0.1, 0.05, 0.025, 0.01, 0.005$ and 0.001.

Table 2a: Critical values of the test with known μ and unknown d.

$\widehat{d}$	$T\backslash\alpha$	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.001
-0.4	50	0.028	0.050	0.086	0.118	0.156	0.220	0.273	0.432
	100	0.028	0.050	0.084	0.115	0.148	0.201	0.246	0.384
	200	0.027	0.048	0.080	0.107	0.138	0.183	0.216	0.307
	500	0.026	0.047	0.077	0.102	0.129	0.168	0.201	0.290
	2000	0.026	0.045	0.074	0.098	0.123	0.156	0.182	0.248
-0.3	50	0.032	0.056	0.095	0.130	0.173	0.241	0.307	0.493
	100	0.032	0.057	0.095	0.127	0.164	0.221	0.273	0.425
	200	0.032	0.057	0.091	0.121	0.154	0.201	0.242	0.349
	500	0.032	0.056	0.091	0.118	0.148	0.188	0.219	0.298
	2000	0.033	0.057	0.091	0.119	0.147	0.188	0.218	0.291
-0.2	50	0.036	0.063	0.105	0.140	0.183	0.255	0.323	0.537
	100	0.038	0.065	0.105	0.138	0.175	0.231	0.282	0.428
	200	0.038	0.066	0.105	0.138	0.172	0.220	0.258	0.355
	500	0.039	0.067	0.106	0.138	0.171	0.215	0.248	0.329
	2000	0.039	0.067	0.106	0.138	0.171	0.215	0.248	0.335
-0.1	50	0.041	0.070	0.112	0.147	0.187	0.247	0.306	0.504
	100	0.043	0.072	0.115	0.149	0.186	0.237	0.279	0.411
	200	0.044	0.074	0.116	0.150	0.186	0.235	0.272	0.355
	500	0.045	0.075	0.118	0.153	0.187	0.238	0.275	0.364
	2000	0.045	0.075	0.118	0.152	0.189	0.239	0.278	0.362

Table 2a (continued)

$\widehat{d}$	$T\backslash\alpha$	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.001
0.0	50	0.042	0.072	0.115	0.151	0.189	0.247	0.297	0.459
	100	0.045	0.075	0.118	0.153	0.189	0.242	0.282	0.374
	200	0.046	0.077	0.121	0.156	0.193	0.242	0.282	0.367
	500	0.046	0.078	0.123	0.159	0.195	0.246	0.287	0.388
	2000	0.047	0.079	0.123	0.159	0.196	0.248	0.286	0.382
0.1	50	0.038	0.068	0.111	0.146	0.184	0.241	0.292	0.451
	100	0.039	0.068	0.110	0.145	0.181	0.231	0.274	0.380
	200	0.039	0.069	0.111	0.146	0.182	0.230	0.267	0.358
	500	0.040	0.069	0.113	0.148	0.185	0.235	0.274	0.360
	2000	0.040	0.070	0.115	0.151	0.188	0.240	0.277	0.379
0.2	50	0.021	0.042	0.074	0.101	0.132	0.179	0.225	0.335
	100	0.018	0.038	0.069	0.096	0.125	0.169	0.207	0.324
	200	0.016	0.035	0.065	0.092	0.121	0.163	0.200	0.294
	500	0.015	0.032	0.062	0.088	0.116	0.156	0.187	0.270
	2000	0.013	0.030	0.059	0.084	0.112	0.151	0.179	0.259

Table 2b: Critical values of the test with unknown μ and d.

$\widehat{d}$	$T\backslash\alpha$	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.001
-0.4	50	0.028	0.050	0.086	0.119	0.156	0.216	0.275	0.424
	100	0.027	0.050	0.083	0.112	0.146	0.198	0.246	0.358
	200	0.027	0.049	0.081	0.108	0.139	0.185	0.225	0.327
	500	0.026	0.047	0.077	0.102	0.130	0.169	0.202	0.282
	2000	0.026	0.046	0.074	0.097	0.123	0.157	0.184	0.246
-0.3	50	0.032	0.057	0.095	0.130	0.171	0.241	0.301	0.493
	100	0.032	0.057	0.094	0.127	0.163	0.221	0.275	0.408
	200	0.032	0.056	0.092	0.122	0.155	0.202	0.243	0.352
	500	0.032	0.056	0.091	0.119	0.148	0.190	0.222	0.297
	2000	0.032	0.056	0.090	0.118	0.147	0.188	0.219	0.292
-0.2	50	0.036	0.063	0.105	0.142	0.185	0.259	0.330	0.538
	100	0.039	0.065	0.105	0.139	0.176	0.231	0.280	0.448
	200	0.039	0.066	0.105	0.137	0.170	0.219	0.259	0.360
	500	0.039	0.067	0.107	0.138	0.172	0.215	0.252	0.338
	2000	0.040	0.067	0.107	0.139	0.173	0.218	0.253	0.334
-0.1	50	0.041	0.069	0.112	0.147	0.188	0.255	0.317	0.523
	100	0.043	0.072	0.114	0.148	0.185	0.235	0.275	0.398
	200	0.044	0.074	0.116	0.150	0.186	0.234	0.271	0.360
	500	0.045	0.075	0.118	0.153	0.189	0.237	0.275	0.363
	2000	0.045	0.075	0.119	0.153	0.189	0.239	0.281	0.368

Table 2b (continued)

$\widehat{d}$	$T\backslash\alpha$	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.001
0.0	50	0.043	0.073	0.117	0.153	0.191	0.248	0.301	0.474
	100	0.045	0.075	0.118	0.154	0.191	0.242	0.283	0.383
	200	0.046	0.077	0.121	0.156	0.192	0.241	0.279	0.367
	500	0.046	0.078	0.122	0.157	0.194	0.244	0.281	0.375
	2000	0.047	0.078	0.123	0.159	0.197	0.247	0.290	0.383
0.1	50	0.042	0.073	0.117	0.153	0.190	0.250	0.304	0.463
	100	0.041	0.071	0.113	0.148	0.181	0.228	0.265	0.361
	200	0.040	0.070	0.113	0.147	0.182	0.230	0.268	0.354
	500	0.040	0.070	0.113	0.148	0.185	0.233	0.270	0.356
	2000	0.040	0.070	0.115	0.151	0.188	0.238	0.279	0.372
0.2	50	0.026	0.050	0.083	0.108	0.136	0.177	0.212	0.321
	100	0.021	0.044	0.077	0.103	0.129	0.169	0.199	0.273
	200	0.018	0.039	0.071	0.097	0.125	0.162	0.186	0.262
	500	0.016	0.035	0.065	0.091	0.119	0.155	0.185	0.257
	2000	0.014	0.032	0.061	0.087	0.115	0.153	0.184	0.251

Experiment 3. This experiment examines the size and the power of the test for $H_0 : y_t \sim I(d)$. Let u_t be i.i.d. $N(0,1)$ random variable, we examine the power of the test against different alternatives for various sample sizes with 10000 replications. Table 3 reports the rejection rates of the test under various alternatives by employing the 5% critical values in Table 2b.

Table 3: The size and the power of the test

True DGP \ T		50	100	200	500
1	$y_t = u_t$	0.048	0.042	0.052	0.054
2	$y_t = I(-0.1)$	0.082	0.059	0.049	0.053
3	$y_t = I(0.2)$	0.092	0.068	0.052	0.046
4	$y_t = -0.5y_{t-1} + u_t$	0.912	0.975	0.997	1.000
5	$y_t = 0.5y_{t-1} + u_t$	0.759	0.903	0.985	1.000
6	$y_t = -.2y_{t-1} + 0.2y_{t-2} + u_t$	0.509	0.713	0.952	1.000
7	$y_t = 0.5y_{t-1} - 0.5y_{t-2} + u_t$	0.637	0.703	0.723	0.750
8	$y_t = u_t + 0.5u_{t-1}$	0.600	0.689	0.869	0.965
9	$y_t = u_t - 0.5u_{t-1}$	0.767	0.856	0.927	0.993
10	$y_t = u_t + 0.5u_{t-1} - 0.5u_{t-2}$	0.636	0.934	0.997	1.000
11	$y_t = u_t - 0.2u_{t-1} + 0.2u_{t-2}$	0.385	0.478	0.781	0.998
12	$y_t = 0.5y_{t-1} + u_t + 0.2u_{t-1}$	0.953	0.998	1.000	1.000
13	$y_t = -0.2y_{t-1} + u_t - 0.5u_{t-1}$	0.967	0.998	1.000	1.000
14	$y_t = 0.2y_{t-1} - 0.2y_{t-2} + u_t - 0.2u_{t-1}$	0.214	0.460	0.734	0.981

Observed from Table 3 that, for the first three cases, the size of the test is approximately equal to 5% when sample size is large. For other cases, as the sample size increases, the power converges to 1. Thus, the test performs well against a wide range of alternatives.

4. Conclusions

Most of the tests for fractional integration in the literature are time-domain tests that are nested within the fractional alternative. This paper develops a frequency-domain based test for fractional integration versus non-fractional integration. Finite-sample distributions of the test are simulated and the corresponding critical values are reported. Our test is shown to have power against a wide variety of alternatives. The applications of our frequency-domain test are extensive. For example, apart from testing for fractional integration, it can also be used to test if two time series data are coming from the same data generating process. In particular, by splitting the data series into two subsamples, our test can also be used to test if there is a structural change in a time series (Chong, 2001; Bai et al. 2008; Hinich and Farley, 1970) by examining if they spectral distribution is different across subsamples.

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