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A Test for a Shifting Slope Coefficient in a Linear Model

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A locally most powerful test is developed for the hypothesis that a slope coefficient in a linear time series model is stable, against the alternative that the slope shifts exactly once somewhere in the series. Analysis of the procedure using artificial data indicates good power characteristics even when the ratio of the shift size to the error variance is moderate—especially if the shift does not occur very near either end of the series. Power also depends on the pattern of the independent variables and on whether the error variance is known or must be estimated using the residuals about the regression line.

1. INTRODUCTION

Statistical apparatus needed to deal with potential slope non-stationarity in a linear time series model depends on the nature of the potential instability. When the exact location of a shift is hypothesized, weighted squared residuals from two regressions (pre- and post-shift) can be compared against squared residuals from a pooled regression to test whether a statistically significant shift occurred at that point [2]. The opposite extreme is the case of a shift which, if it occurs at all, does so at an unknown point. One attack on the latter problem [6, 7] involves calculating regressions for observations $(1, \cdots, i+k)$ and $(i+k+1, \cdots, n)$ where k parameters are to be estimated and i ranges from 1 to $n-2k$. Choosing i^* as the dividing point which maximizes the joint likelihood of the two regressions, a chi-square test might be used on a likelihood ratio as an approximate test as to whether the parameters of a pooled regression differ from those of the two sub-regressions. This iterative procedure has shortcomings because the likelihood function is not differentiable with respect to the parameter i^*, because of power difficulties and because the distribution of $-2\log\lambda$ deviates markedly from a chi-square distribution under the null hypothesis [7].

This article develops a test of the null hypothesis that a slope coefficient in a time series model does not shift, against the alternative that the parameter shifts exactly once and the potential shift is small relative to the error variance. This formulation is convenient because it takes advantage of the ratio of the shift size to the standard deviation [3], much as the known shift point formulation mentioned previously takes advantage of certain characteristics of that problem. Further, many problems involving uncertainty about presence as well as the position of a shift are also likely to involve shifts which are small relative to sampling error.

The simple, easy test requires only the inputs, parameter estimates, and residuals from an ordinary regression. It can readily be incorporated into a

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regression routine, much as the Durbin-Watson test is often used to flag problems involving serial correlation in the errors. When a shift is indicated, the Quandt procedure can be used to estimate its location.

2. LINEAR MODEL WITH A RANDOM PARAMETER SHIFT

The basic model is

$$Y_k = \alpha_k + \gamma_k X_k + \epsilon_k \qquad k = 1, \cdots, t \tag{2.1a}$$

where X_k is measured without error. The ϵ_k are independent normal zero mean random variables which have equal variances, i.e., $\{\epsilon_1, \cdots, \epsilon_t\} \sim N(0, \sigma^2 I)$.

$$\gamma_k = \begin{cases} \gamma \text{ if } 1 < k \leq T \\ \gamma + \delta \text{ if } T \leq k \leq t \end{cases} \qquad \alpha_k = \begin{cases} \alpha \text{ if } 1 \leq k < T. \\ \alpha - \delta X_T \text{ if } T \leq k \leq t. \end{cases} \tag{2.1b}$$

The shift point, T, is a random variable which is assumed to have the rectangular distribution

$$P(T = t^*) = (1/t) \qquad t^* = 1, \cdots, t. \tag{2.2}$$

That is, given that a shift occurs during the observation period, each time t^* is equally likely to be the shift point. Having no *a priori* knowledge about shift points corresponds to the situation in which the investigator wishes to make a very general test as to whether his linear model, specified as such models generally are as having a single pair of parameters over the entire period, is consistent with the data or whether it is not. Other tests and models may involve different likelihoods. For example, the Chow test mentioned earlier is designed for quite different purposes (finding whether a shift occurs at some pre-hypothesized point), and the underlying specification of (2.2) and the test procedure are quite different.

Under the null hypothesis,

$$E_0(\boldsymbol{Y}) = \alpha \boldsymbol{1} + \gamma \boldsymbol{X}, \tag{2.3}$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are column vectors of observations and $\boldsymbol{1}$ is the column vector of ones.

If a shift of size δ occurs when $T = t^*$, then

$$E_\delta(\boldsymbol{Y} \mid T = t^*) = \alpha \boldsymbol{1} - \gamma \boldsymbol{X} + \delta \boldsymbol{Z}(t^*) \tag{2.4}$$

where

$$\boldsymbol{Z}(t^*)' = (0, \cdots, 0, X_{t*+1} - X_{t*}, \cdots, X_t - X_{t*}).$$

Using (2.2),

$$\begin{aligned} E_\delta(\boldsymbol{Y}) &= \frac{1}{t} \sum_{s=1}^{t} E_\delta(\boldsymbol{Y} \mid t^* = s) \\ &= \alpha \boldsymbol{1} + \gamma \boldsymbol{Y} + \frac{\delta}{t} \sum_{s=1}^{t} \boldsymbol{Z}(s). \end{aligned} \tag{2.5}$$

Defining

$$\boldsymbol{\theta}' = \left(0, \frac{1}{t}\sum_{j=1}^{2}(X_2 - X_j), \cdots, \frac{1}{t}\sum_{j=1}^{t}(X_t - X_j)\right)$$

$$E_\delta(\boldsymbol{Y}) = \alpha\mathbf{1} + \gamma\boldsymbol{X} + \delta\boldsymbol{\theta}. \tag{2.6}$$

Under H_0, the density function of Y is

$$f_0(y) = (2\pi\sigma^2)^{-t/2}\exp\left[-\frac{1}{2\sigma^2}(y - \boldsymbol{\mu})'(y - \boldsymbol{\mu})\right] \tag{2.7}$$

where $\mu = \alpha\mathbf{1} + \gamma\boldsymbol{X}$. Under H_δ,

$$f_\delta(y) = (2\pi\sigma^2)^{-t/2}\sum_{s=1}^{t} t^{-1}\exp\left[-\frac{1}{2\sigma^2}(y - \boldsymbol{\mu} - \delta\boldsymbol{Z}(s))'(y - \boldsymbol{\mu} - \delta\boldsymbol{Z}(s))\right]. \tag{2.8}$$

The likelihood ratio, $f_\delta(y)/f_0(y)$, is then

$$\lambda(y \mid \alpha, \gamma, \delta) = \sum_{s=1}^{t} t^{-1}\exp\left[\frac{\delta}{\sigma^2}(y - \boldsymbol{\mu})'\boldsymbol{Z}(s) - \frac{\delta^2}{2\sigma^2}\boldsymbol{Z}(s)'\boldsymbol{Z}(s)\right]. \tag{2.9}$$

Expanding λ about $\delta = 0$,

$$\lambda = 1 + \sum_{s=1}^{t} t^{-1}\frac{\delta}{\sigma^2}(y - \boldsymbol{\mu})'\dot{\boldsymbol{Z}}(s) + O(\delta^2) \tag{2.10}$$

$$= 1 + \frac{\delta}{\sigma^2}(y - \boldsymbol{\mu})'\boldsymbol{\theta} + O(\delta^2). \tag{2.11}$$

Reversing $\boldsymbol{\theta}$ and $y - \boldsymbol{\mu}$ in (2.11), the first order approximation of the likelihood ratio test is a test which rejects H_0 if

$$S = \boldsymbol{\theta}'(\boldsymbol{Y} - \alpha\mathbf{1} - \gamma\boldsymbol{X}) \tag{2.12}$$

is significantly different from zero. For fixed a, γ, δ, and σ^2 the preceding test is locally most powerful as $\delta \rightarrow 0$ [5]. Capon [1] has shown that under certain regularity conditions for the population distribution functions, which are satisfied by the previous H_0 and H_δ distributions, the locally most powerful test is asymptotically efficient as compared to the likelihood ratio test. A random sample from the H_0 or H_δ population consists of a set of independently and identically distributed vectors $\boldsymbol{Y}^{(1)}, \cdots, \boldsymbol{Y}^{(n)}$. In most applications only one sample vector is available, but (2.12) can still be used in a test with fixed probability of Type I error.

The parameters α and γ are usually unknown and must be estimated. A manageable procedure is to replace α and γ in (2.12) with their least squares estimates under the null hypothesis, i.e., the test statistic becomes

$$\tilde{S} = \boldsymbol{\theta}'(\boldsymbol{Y} - \hat{\alpha}\mathbf{1} - \hat{\gamma}\boldsymbol{X}) = \boldsymbol{\theta}'(I - A)\boldsymbol{Y}$$

where

$$A = (1/t)\mathbf{1}\,\mathbf{1}' + \frac{1}{\boldsymbol{X}'\boldsymbol{X} - t\overline{\boldsymbol{X}}^2}(\boldsymbol{X} - \overline{\boldsymbol{X}})'(\boldsymbol{X} - \overline{\boldsymbol{X}}). \tag{2.13}$$

and $\overline{\boldsymbol{X}}$ is a vector with the arithmetic mean of the explanatory variable, $\overline{\boldsymbol{X}}=(1/t)\boldsymbol{X}'\mathbf{1}$, as each element. The matrix A is the projection onto the subspace of Euclidean t-space spanned by $\mathbf{1}$ and $\boldsymbol{X}$. Thus if θ is in this subspace, the test statistic $\tilde{S}$ is identically 0 for any Y. However θ is in the subspace if and only if

$$kX_k - \sum_{j=1}^{k} X_j = a + bX_k \qquad k = 1, \cdots, t \tag{2.14}$$

for some scalars a and b. Unless $X_2 = \cdots = X_t$, there are no a and b such that (2.14) holds for each $k=1, \cdots, t>2$ and thus θ is not in the subspace.

Since $\hat{\alpha}$ and $\hat{\gamma}$ are unbiased under H_0,

$$E_0(\tilde{S}) = 0. \tag{2.15}$$

Under the alternative hypothesis, using (2.10) and the definitions of α and $\hat{\gamma}$,

$$E_\delta(\hat{\gamma}) = \gamma + \delta \frac{\boldsymbol{X}'\boldsymbol{\theta} - t\overline{X}\overline{\theta}}{\boldsymbol{X}'\boldsymbol{X} - t\overline{X}^2} \tag{2.16}$$

$$E_\delta(\hat{\alpha}) = \alpha + \delta\left(\overline{\theta} - \left[\frac{\boldsymbol{X}'\boldsymbol{\theta} - t\overline{X}\overline{\theta}}{\boldsymbol{X}'\boldsymbol{X} - t\overline{X}^2}\right]\overline{X}\right) \tag{2.17}$$

where $\overline{\theta}=(1/t)(\boldsymbol{\theta}'\mathbf{1})$. From (2.16) and (2.17), the expected value of the residual vector under H_δ is

$$E_\delta(\boldsymbol{Y} - \hat{\alpha}\mathbf{1} - \hat{\gamma}\boldsymbol{X}) = \delta(I - A)\boldsymbol{\theta}. \tag{2.18}$$

The covariance matrices of the residuals are

$$\sigma^{-2}COV_0(\boldsymbol{Y} - \hat{\alpha}\mathbf{1} - \hat{\gamma}\boldsymbol{X}) = I - A \tag{2.19a}$$

and

$$\sigma^{-2}COV_\delta(\boldsymbol{Y} - \hat{\alpha}\mathbf{1} - \hat{\gamma}\boldsymbol{X}) = I - A + \left(\frac{\delta}{\sigma}\right)^2 (I - A)B(I - A), \tag{2.19b}$$

where

$$B = (1/t)\sum_{s=1}^{t} \boldsymbol{Z}(s)\boldsymbol{Z}(s)' - \boldsymbol{\theta}\boldsymbol{\theta}'.$$

Applying (2.13) and (2.19) to (2.12) with α and γ replaced by their least-squares estimates under H_0 yields

$$E_\delta(\tilde{S}) = \delta\boldsymbol{\theta}'(I - A)\boldsymbol{\theta}. \tag{2.20}$$

The variances of $\tilde{S}$ are:

$$\sigma^{-2}V_0(\tilde{S}) = \boldsymbol{\theta}'(I - A)\boldsymbol{\theta} \tag{2.21a}$$

and

$$\sigma^{-2}V_\delta(\tilde{S}) = \boldsymbol{\theta}'(I - A)\boldsymbol{\theta} + \left(\frac{\delta}{\sigma}\right)\left\{\frac{1}{t}\sum_{s=1}^{t}[\boldsymbol{\theta}'(I - A)\boldsymbol{Z}(s)]^2 - [\boldsymbol{\theta}'(I - A)\boldsymbol{\theta}]^2\right\} \tag{2.21b}$$

$$= \sigma^{-2}V_0(\tilde{S}) + O\left(\frac{\delta^2}{\sigma^2}\right),$$

respectively. Under H_0, $\tilde{S}$ is normally distributed with mean zero and variance $V_0(\tilde{S})$. Given a shift at $T=t^*$, $\tilde{S}$ is normally distributed with mean $\delta\theta'(I-A)Z(t^*)$ and variance $V_0(\tilde{S})$. Thus under H_δ, the distribution of $\tilde{S}$ is an average over t^* of these normal distribution functions, which in general is multimodal. Given a Type I error of α, the null hypothesis is rejected if $|\tilde{S}|$ is greater than $z_{\alpha/2}\sigma\sqrt{\boldsymbol{\theta}'(I-A)\boldsymbol{\theta}}$ where $z_{\alpha/2}$ is the value such that $Pr(Y>z_{\alpha/2})=\alpha$ for Y distributed $N(0, 1)$.

The detectability D_δ of a shift in γ is defined as the ratio of the absolute difference between the expected values of $\tilde{S}$ under H_δ and H_0, to the standard deviation of $\tilde{S}$, i.e.,

$$D_\delta = \frac{|E_\delta(\tilde{S}) - E_0(\tilde{S})|}{\sqrt{V_0(\tilde{S})}} = \frac{\delta}{\sigma}\left[(\boldsymbol{\theta}'\boldsymbol{\theta} - t\bar{\theta}^2) - \frac{(\boldsymbol{X}'\boldsymbol{\theta} - t\bar{X}\bar{\theta})^2}{\boldsymbol{X}'\boldsymbol{X} - t\bar{X}^2}\right]^{1/2}. \tag{2.22}$$

(Note that using $V_\delta(\tilde{S})$ in the denominator of (2.22) would produce $D_\delta+O(\delta^2/\sigma^2)$. Given a critical region for $\tilde{S}$, the detectability is directly related to the power of the test for a given small δ—the greater the detectability, the greater the power and vice versa; the detectability is thus a convenient measure of the power of the test for δ in the neighborhood of zero.

The statistic $\tilde{S}$ has the maximum detectability of any linear combination of the residuals. This can be shown as follows: Define a statistic S^*,

$$S^* = \varphi'(Y - \hat{\alpha}\mathbf{1} - \hat{\gamma}\boldsymbol{X}) \tag{2.23}$$

for a vector of constants $\varphi' = (\varphi_1, \cdots, \varphi_t)$ which is not a linear combination of $\mathbf{1}$ and $\boldsymbol{X}$. From (2.13) and (2.18),

$$E_\delta(S^*) = \delta\varphi'(I - A)\boldsymbol{\theta} \tag{2.24}$$

and

$$V_0(S^*) = \sigma^2\varphi'(I - A)\varphi. \tag{2.25}$$

Thus the detectability $D_\delta{}^*$ of a shift in γ using S^* is

$$D_\delta{}^* = \frac{\delta}{\sigma}\frac{|\varphi'(I - A)\varphi|}{\sqrt{\varphi'(I - A)\varphi}}. \tag{2.26}$$

By the Schwarz inequality,

$$D_\delta{}^* \leq D_\delta \tag{2.27}$$

with the equality holding if and only if $\varphi = c\boldsymbol{\theta}+\boldsymbol{\psi}$ where $\boldsymbol{\psi}$ is any vector satisfying $(I-A)\boldsymbol{\psi}=0$. Thus $\tilde{S}$ has the maximum detectability of all linear combinations of the residuals.

We have assumed up to now that σ^2 is known. In most applications the error variance is unknown, but can be estimated from the residuals using the usual estimator of σ^2, i.e.,

$$s^2 = \frac{1}{t-2} \sum_{k=1}^{t} (Y_k - \hat{\alpha} - \hat{\gamma} X_k)^2$$
$$= \frac{1}{t-2} Y'(I - A)Y. \tag{2.28}$$

Under H_0, $(t-2)s^2/\sigma^2$ has a chi-square distribution with $t-2$ degrees of freedom. Given a shift at $T=t^*$, $(t-2)s^2/\sigma^2$ has a non-central chi-square distribution with non-centrality parameter

$$\nu = (1/2)\delta^2 \boldsymbol{Z}(t^*)'(I - A)\boldsymbol{Z}(t^*)$$
$$\leq (1/2)\delta^2 \sum_{k=t^*+1}^{t} (X_k - X_{t^*})^2. \tag{2.29}$$

Thus under H_0, $E_0(s^2) = \sigma^2$ and under H_δ, the expected value of s^2 satisfies the inequality

$$0 \leq E_\delta(s^2) - \sigma^2 \leq \delta^2[\max_{j,k} (X_k - X_j)^2]. \tag{2.30}$$

The null hypothesis is rejected if $|\tilde{S}|$ is greater than $z_{\alpha/2} s\sqrt{\boldsymbol{\theta}'(I-A)\boldsymbol{\theta}}$. Since $\tilde{S}$ and s^2 are correlated even under the null hypothesis, the distribution of $\tilde{S}/s$ is not a standard well-known distribution like Student's t. However, under H_0, $s^2 \rightarrow \sigma^2$ in probability as $t \rightarrow \infty$ and thus

$$Pr\left[\frac{|\tilde{S}|}{s\sqrt{\boldsymbol{\theta}'(I - A)\boldsymbol{\theta}}} > z_{\alpha/2}\right] \rightarrow \alpha.$$

Moreover, the conditional power of this test is less than or equal to the conditional power of the test where σ^2 is known since from (2.29) and (2.30), $s^2/\sigma^2 \rightarrow 1 + O[(\delta/\sigma)^2]$ as $t \rightarrow \infty$ where $O[(\delta/\sigma)^2] \geq 0$.

3. POWER AND DISCRIMINATION

Information about small-sample properties of the test are obtained most conveniently through computer analysis using artificial data. For example, the fact that the test has maximum detectability does not necessarily give a clear picture of its performance. For one thing, the development is based on approximations and limiting properties which may produce procedures which are not powerful against certain classes of alternative hypotheses [3]. For another, as can be seen in (2.22), the power depends on both the size of the shift and the pattern of independent variables. This systematic relationship between power and configuration of explanatory variables requires explicit investigation for any application of the procedure. It may sometimes be convenient to study analytical properties of the procedure for specific patterns of independent variables but Monte Carlo analysis is a more general approach, since the same computer program can be used to analyze any model like (2.1) and any pattern of independent variables.

In this case, Monte Carlo tests of the procedure just developed were conducted on the equation used by Quandt [7] in a similar analysis:

$$Y_k = 2.5 - .7X_k + \epsilon_k \qquad k = 1, \cdots, 64. \tag{3.1}$$

For the present test, the error terms, $N(0, 1)$, were generated by the Box-Muller transformation on a sequence of pseudo-random rectangular variables [4]. The experiment was repeated on three different patterns of explanatory variables (the X's), because of the detectability depending on the patterns of the explanatory variables (2.22).

1. *Rectangular:* the twenty values of X used by Quandt [7] in his experiments were supplemented by 44 other numbers distributed rectangularly between 0 and 20, which were drawn from a table of random digits.
2. *Linear trend up in the explanatory variables:* $X_1=1, X_2=2, \cdots, X_{64}=64$.
3. *Linear trend down in the explanatory variables:* $X_1=64, X_2=63, \cdots, X_{64}=1$.

For each of the three patterns of independent variables, shifts ranging in size between $\sigma=-2.$ to $\sigma=2.$ were used in increments of .1 between these two values. For each of these 39 values of δ, 250 separate series of residual errors (the ϵ's in (3.1)) were generated. The shifts were positioned rectangularly over the 64 data points for each value of δ resulting in the following configuration of the experiment in terms of the model expressed in (2.1a), (2.1b) and (3.1)

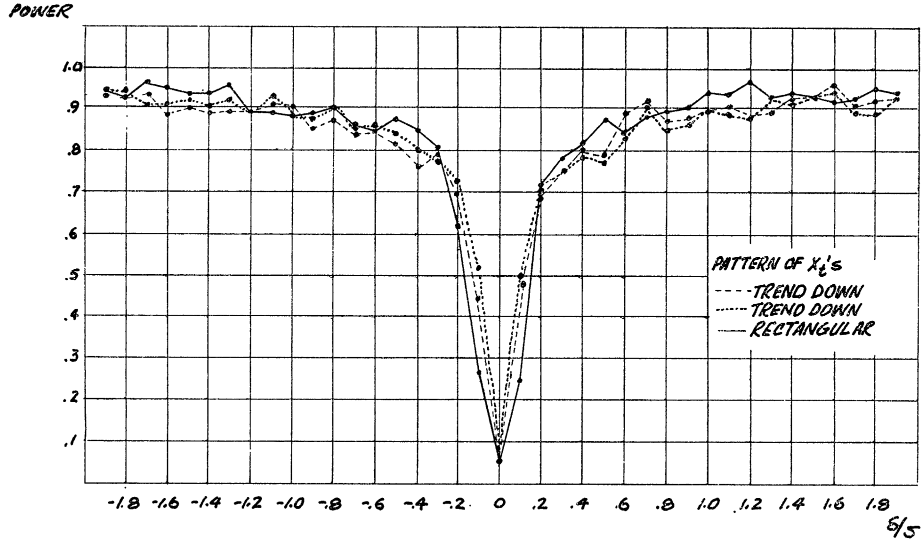
$$\boldsymbol{Y} = 2.51 - .7\boldsymbol{X} + \delta\boldsymbol{Z}(t^*) + \epsilon \tag{3.2}$$

where δ ranges from -2 to $+2$ by increments of .1, and $\boldsymbol{Z}(t^*)' = (0, \cdots, 0, X_{t^*+1}-X_{t^*}, \cdots, X_t - X_{t^*})$, with t^* distributed rectangularly over $t=1, \cdots, 64$. Further, a full set of such Monte Carlo analyses was carried out under each of two conditions: (1) with the variance of ϵ_k known and (2) with this variance estimated using (2.28).

3.1 Analysis with σ known

Plots of the fraction of null (no shift) hypotheses rejected at $\alpha=.05$ are shown in Figure 1 for each pattern of explanatory variable—rectangular, and linear trends up and down. As was expected from (2.22), the estimated power depends on the configuration of the independent variables, although the discrepancy is not great for these particular patterns except for very small values of δ/σ. Under the experiments, the probabilities of rejecting the null hypothesis are not one for even large values of δ/σ because the discriminatory power also depends on the position in the series at which a shift occurs. It is reasonable, for example, to expect that a shift which occurs either very early or very late in a series will be harder to detect than one which occurs in the middle range. If the shift occurs in the middle of the series, enough observations are available under both regimes to make comparison feasible. By contrast, a shift which occurs between the first and second observation is extremely difficult to detect with a general-purpose procedure, unless the shift itself is enormous relative to the standard error of the residuals. Figure 2 for example, shows a slightly smoothed plot of the estimated power related to positions at which shifts occurred for three shift sizes ($\delta=-.2$, $-.5$, and -1.0) in series with independent variables distributed rectangularly. For these trials, the test discriminates perfectly for shifts in the middle range, even for small δ, and the

Figure 1. ESTIMATED POWER FOR ALTERNATIVE PATTERNS OF INDEPENDENT VARIABLES WHEN VARIANCE IS KNOWN



range of perfect discrimination broadens as the shift becomes larger. For example, when the size of the parameter shift equalled the standard deviation of the error term, the test failed to discriminate perfectly in the experiment only in the first five and last five positions of the 64-observation series. Complete failure to detect a parametric shift occurred only in the first four and last two positions. Failure to detect a shift early in a series has relatively little effect on either the accuracy of a short-run forecast or the accuracy of the estimated coefficients in (3.2); failure to detect shifts at the very end of a series may have substantial effect on both the forecast and the estimated parameters in terms of applications.

Shifts near the ends of a series are very difficult to handle in the context of a general-purpose test such as this, and are beyond the scope of this article.

Figure 2. ESTIMATED POWER AS A FUNCTION OF SHIFT POSITION AND SHIFT SIZE: RECTANGULAR INDEPENDENT VARIABLES, KNOWN VARIANCE

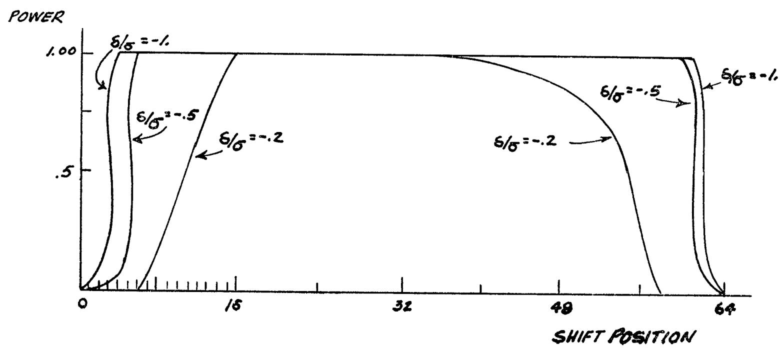
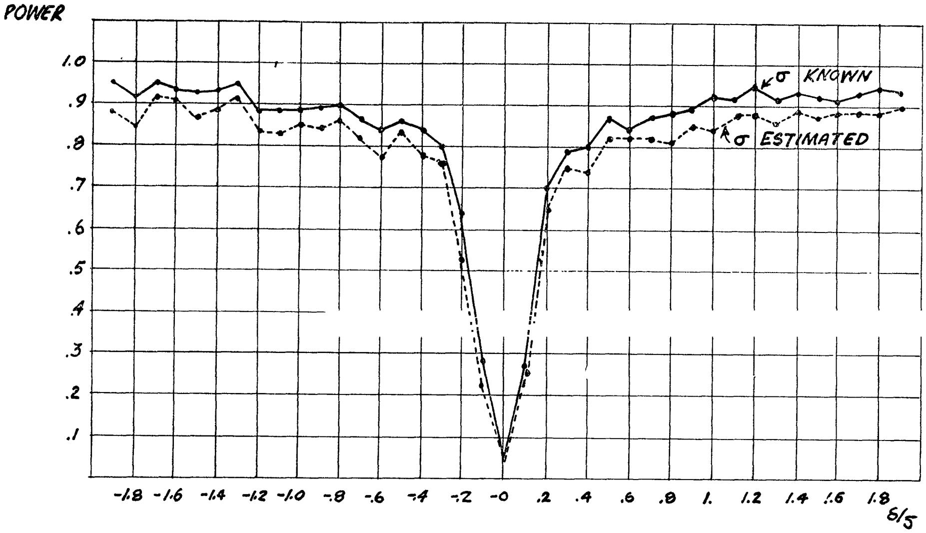


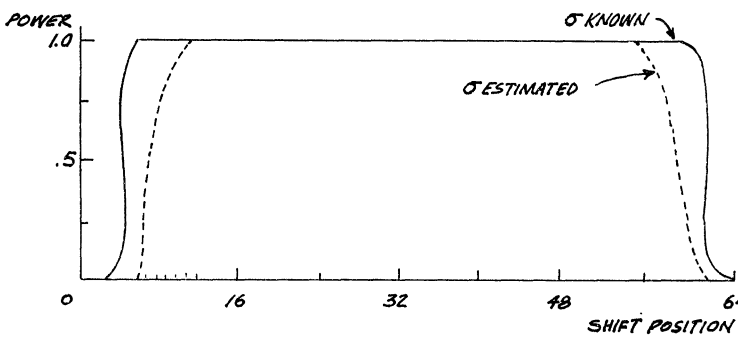
Figure 3. ESTIMATED POWER FOR KNOWN VERSUS ESTIMATED VARIANCE: RECTANGULAR INDEPENDENT VARIABLES



3.2 Analysis with σ estimated

The power is affected by replacing σ with $\hat{\sigma}$ estimated with the residuals about the regression line as in (2.28), but the general pattern of performance is similar to that when σ is known. For example, Figure 3 shows estimated power for σ known and estimated with $\hat{\sigma}$, using rectangular independent variables. Estimation generally reduces power about five percent for large values of δ/σ and by smaller amounts for smaller shift-to-error ratios. The reduction in power occurs, as before, in failure to detect parameter shift early and late in the series. For example, Figure 4 shows estimated power, again smoothed slightly, plotted against shift position for $\delta/\sigma=-.5$ for both cases. Again, detection is perfect in the middle range of the series in both cases, but this range

Figure 4. ESTIMATED POWER AS A FUNCTION OF SHIFT POSITION FOR KNOWN VERSUS ESTIMATED VARIANCE: RECTANGULAR INDEPENDENT VARIABLES, $\delta=-.5$



is narrower when the variance is estimated than when it is known. Power is effected similarly in the cases of the other patterns of independent variables.

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