Maximum-likelihood signal processing for a vertical array*

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(Received 7 July 1972)

This paper presents the maximum-likelihood signal processor for steering a vertical array in the vertical direction. The major application is to the estimation of the depth of a distant narrow-band continuous point source in the waveguide. The eigenfunctions of the guide are used to match the array to the received signal. The error of the depth estimate is derived as a function of the aperture and geometry of the array, the covariance function of the ambient noise received by the array, and the observation period; assuming that the source and medium are stationary during that period. The processing technique can be applied to any perfect waveguide in which a signal source is detected by an array of sensors.

Subject Classification: 15.2, 15.3.

INTRODUCTION

In a comprehensive review article Clay[1] uses normal mode theory of acoustic waveguides to analyze the signal-to-noise gain of horizontal and vertical hydrophone arrays in a noisy ocean. He shows that horizontal rather than vertical arrays should be used to obtain bearing estimates for a signal source. Vertical arrays should be used to obtain estimates of the vertical wavenumber components and depth parameters of the source. A review of optimal processing of horizontal arrays is given by Clay, Hinich, and Shaman.[2] This paper presents the maximum-likelihood signal processing of a vertical array immersed in any perfect waveguide.

A major application of the technique is the optimal estimation of the depth of a continuous point source in the waveguide. The signal-processing technique was motivated by Clay's suggestion that a vertical array should be matched to the eigenfunctions of the guide, which are not in general sinusoidal functions in the vertical direction. Consequently, except in the simplest guides, the well known delay-and-sum beam steering will not be appropriate for obtaining the source depth.

The acoustical pressure at a hydrophone at depth x due to a continuous wave transmission from a distant narrow-band point source at depth x_0 is the real part of

$$p(x,t)=r^{-\frac{1}{2}}\sum_{m=1}^{M}\frac{a_m}{\gamma_m}\exp[i(\omega t-\kappa_m r+\tfrac{1}{4}\pi)]\varphi_m(x_0)\varphi_m(x), \quad (1)$$

where r is the range and ω is the frequency of the source, φ_m is the mth mode eigenfunction, κ_m and γ_m are respectively the horizontal and vertical components of the mth wavenumber, and a_m is the attenuated excitation of the mth mode (Fig. 1).[3] Suppose that we filter a T sec record of output from the sensor using a filter centered at ω of bandwidth $\partial\omega\approx 1/T$ Hz. If the source and the medium are stationary during the T sec period, the observed amplitude will be proportional to the signal

$$f(x)=r^{-\frac{1}{2}}\sum_{m=1}^{M}\frac{a_m\cos\kappa_m r}{\gamma_m}\varphi_m(x_0)\varphi_m(x)+\epsilon(x), \quad (2)$$

where the variance of the noise $\epsilon(x)$ is *proportional* to $1/T$. Thus the power of the received ambient noise can be significantly reduced if the process is stationary for a sufficiently long observation period. In other words, if the noise at each receiver is phase incoherent when the signal is coherent, time averaging reduces the noise.

The horizontal and vertical components of the mth mode wavenumber are related to ω by the dispersion equation

$$\kappa_m^2+\gamma_m^2=\frac{\omega^2}{c^2}, \quad (3)$$

where c denotes the phase velocity of the wave. If there is no structural dispersion, c is independent of ω, and is equal to the group velocity of energy radiation in the medium.

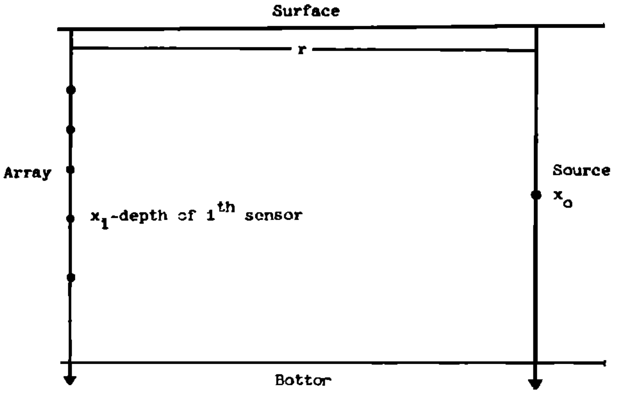

FIG. 1. Vertical array of sensors.

There is no structural dispersion for the simplest model of the ocean as an acoustic waveguide; the homogeneous compressible fluid waveguide with a rigid bottom and a free surface. In the absence of gravity effects the eigenfunctions for this waveguide are $\varphi_m = \sqrt{2}\cos\gamma_m x$, $0 \le x \le D$, where D is the depth of the bottom, and $\gamma_m = (m+\frac{1}{2})\pi/D$. However, owing to surface gravity effects the lower modes reflect off the surface, and thus for small γ_m the eigenfunctions are $\varphi_m = \sqrt{2}\sin(m\pi/D)x$.[1,4]

The steady-state normal mode solution to the idealized acoustic waveguide is a special case of the general solution of the inhomogeneous Sturm–Liouville partial differential equation. The maximum-likelihood estimators in this paper are functions of the eigenfunctions of the S–L operator.

I. STURM-LIOUVILLE PROBLEM

Suppose that at n points in the unit interval [0,1] we observe a noisy version of the solution $f(x)$ to the inhomogeneous Sturm–Liouville problem $Lf = s(x)$ where the operator is defined by

$$Lf = -\frac{\partial}{\partial x}\left(p\frac{\partial}{\partial x}f\right) + qf, \tag{4}$$

$s(x)$ is the intensity of a source at point x, p is a given continuously differentiable positive function of $x \in [0,1]$, q is non-negative and continuous in [0,1], and f satisfies the following boundary conditions given $c_i > 0$, $i = 1,\dots,4$:

$$\begin{aligned} c_1 f(0) - c_2\frac{\partial}{\partial x}f(0) &= 0 \\ c_3 f(1) + c_4\frac{\partial}{\partial x}f(1) &= 0. \end{aligned} \tag{5}$$

For steady-state processes $f(x)$ is the solution to the general *wave* or *diffusion* equation with the given boundary conditions.[5]

It is well known that L is a *positive self-adjoint linear operator* whose eigenvalues $\gamma_1 \le \gamma_2 \le \cdots$ are discrete with finite multiplicity, and whose eigenfunctions $\varphi_1, \varphi_2, \dots$ are complete in L_2, the space of square integrable functions of $x \in [0,1]$. Moreover, assuming $\gamma_1 > 0$ (L is non-singular),

$$f(x) = \sum_{m=1}^{\infty} (s,\varphi_m)\gamma_m^{-1}\varphi_m(x) \tag{6}$$

uniformly in x, where

$$(s,\varphi_m) = \int_0^1 s(x)\varphi_m(x)dx$$

and the eigenfunctions are orthonormal, i.e., $(\varphi_m,\varphi_m) = 1$ and $(\varphi_m,\varphi_{m'}) = 0$ for $m \ne m'$. When L is the wave operator, the φ_m are called the normal modes.

Suppose that the source intensity can be reasonably approximated by a linear combination of the first M eigenfunctions

$$s(x) = \sum_{m=1}^{M} \theta_m\varphi_m(x), \tag{7}$$

where $\theta_1,\dots,\theta_M$ are unknown weights whose values are such that $s(x)$ has a global maximum at $x = x_0$. For the acoustic waveguide discussed in the Introduction, $\theta_m = \varphi_m(x_0)a_m r^{-\frac{1}{2}}\cos\kappa_m r$. If the coefficients $a_m\cos\kappa_m r$ are equal for $m = 1,\dots,M$, then as $M \to \infty$, $s(x) \to \delta(x - x_0)$—the Dirac delta function centered at x_0. Given a set of observations of $y(x) = f(x) + \epsilon(x)$ where $\epsilon(x)$ denotes the noise field at x, we wish to obtain estimates of the θ_m and x_0. For a given application the choice of M must be determined experimentally.[6]

II. ESTIMATING THE SOURCE WEIGHTS

The observations $y(x_i) = f(x_i) + \epsilon(x_i)$ represent the (time) filtered output from a sensor located at point x_i ($i = 1,\dots,n$). The array of sensors lies on a line imbedded in a medium bounded at $x = 0$ and 1. The results in this paper generalize easily to the inhomogeneous boundary value problem involving a self-adjoint operator on $L_2(S)$, the space of square integrable functions of $x \in S$—a closed and bounded subset of Euclidean space.

Applying Eq. 7 to Eq. 6,

$$y(x_i) = \sum_{m=1}^{M} \theta_m\gamma_m^{-1}\varphi_m(x_i) + \epsilon(x_i),$$

or in vector and matrix notation,

$$\mathbf{y} = \mathbf{\Phi}\mathbf{\Gamma}^{-1}\boldsymbol{\theta} + \boldsymbol{\varepsilon},$$

where $\mathbf{y} = [y(x_1),\dots,y(x_n)]'$, $\boldsymbol{\varepsilon} = [\epsilon(x_1),\dots,\epsilon(x_n)]'$, $\boldsymbol{\theta} = (\theta_1,\dots,\theta_M)'$, $\mathbf{\Gamma}$ is the $M \times M$ diagonal matrix whose mth diagonal element is γ_m, and $\mathbf{\Phi}$ is the $n \times M$ matrix whose i, mth element is $\varphi_m(x_i)$.

Let us assume that Φ has column rank M, and that the noise vector $\boldsymbol{\varepsilon}$ has a multivariate Gaussian distribution with zero mean and variance-covariance matrix $\sigma^2\mathbf{\Sigma}$ where σ is an unknown scale parameter. The gradient of the log likelihood is

$$\nabla_\theta \log L(\mathbf{y}|\boldsymbol{\theta}) = \mathbf{\Gamma}^{-1}\mathbf{\Phi}'\mathbf{\Sigma}^{-1}(\mathbf{y} - \mathbf{\Phi}\mathbf{\Gamma}^{-1}\boldsymbol{\theta}). \tag{8}$$

The maximum-likelihood estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is just the generalized least-squares estimator[7,8]

$$\hat{\boldsymbol{\theta}} = \mathbf{\Gamma}(\mathbf{\Phi}'\mathbf{\Sigma}^{-1}\mathbf{\Phi})^{-1}\mathbf{\Phi}'\mathbf{\Sigma}^{-1}\mathbf{y}, \tag{9}$$

whose variance-covariance matrix is

$$E(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})' = \sigma^2\mathbf{\Gamma}(\mathbf{\Phi}'\mathbf{\Sigma}^{-1}\mathbf{\Phi})^{-1}\mathbf{\Gamma}. \tag{10}$$

If the column rank of Φ is less than M, the generalized inverse least-squares estimation technique suggested by Rao[9] should be used.

A simple example would be useful here to illustrate the dependence of the precision of $\hat{\theta}$ on the signal-to-noise ratio and the eigenvalues. Suppose that $\Sigma=I$, the $n\times n$ identity matrix, and that $x_1=0$, $x_n=1$ with n sufficiently large such that

$$\frac{1}{n}\sum_{i=1}^{n}\varphi_m^2(x_i)\approx 1$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\varphi_m(x_i)\varphi_{m'}(x_i)\approx 0,$$

i.e., the ambient noise picked up by the array is assumed to be white, and the sensors are positioned in the waveguide such that the sampled eigenfunctions are approximately orthogonal in Euclidean n-space. Both assumptions are unrealistic for the ocean waveguide since the noise is highly coherent in the vertical direction,[3,10] and it is extremely difficult to design hydrophones to operate at great depths.

From Eqs. 9 and 10 it follows that the maximum-likelihood estimator (least-squares) estimator of θ_m is

$$\hat{\theta}_m=\gamma_m\sum_{i=1}^{n}\varphi_m(x_i)y(x_i), \tag{11}$$

whose variance is

$$E(\hat{\theta}_m-\theta_m)^2=\sigma^2\gamma_m^2/n, \tag{12}$$

and $\hat{\theta}_m$ and $\hat{\theta}_{m'}$ are uncorrelated for $m\neq m'$. Thus the *root mean square proportional error* of $\hat{\theta}_m$ is $\gamma_m(\sigma/\theta_m n^{\frac{1}{2}})$.

III. ESTIMATING THE SOURCE DEPTH x_0

The maximum-likelihood estimator (MLE) of the source function is simply

$$\hat{s}(x)=\sum_{m=1}^{M}\hat{\theta}_m\varphi_m(x).$$

However, the maximum-likelihood estimator of x_0 is a nonlinear function of the observations which can not be easily implemented in practice. Starting with a simple estimator of x_0, we will use the method of scoring[9] to approximate the MLE for small $\sigma n^{-\frac{1}{2}}$.

The simple estimator of x_0 is the point in the unit interval which maximizes $\hat{s}(x)$, i.e.,

$$\hat{s}(\hat{x})=\max_{0\leq x\leq 1}\hat{s}(x). \tag{13}$$

In order to compute the mean squared error of $\hat{x}$, we will need the variance of $(\partial/\partial x)e(x)$, where $e(x)=\hat{s}(x)-s(x)$. Setting $\boldsymbol{\varphi}(x)=[\varphi_1(x),\ldots,\varphi_M(x)]'$, and $\boldsymbol{\varphi}_x(x)=[(\partial/\partial x)\varphi_1(x),\ldots,(\partial/\partial x)\varphi_M(x)]'$ it follows from Eq. 10 that this variance is

$$E\left[\frac{\partial}{\partial x}e(x)\right]^2=\sigma^2\boldsymbol{\varphi}_x'(x)\boldsymbol{\Gamma}(\boldsymbol{\Phi}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Phi})^{-1}\boldsymbol{\Gamma}\boldsymbol{\varphi}_x(x). \tag{14}$$

The rms error of $\hat{x}$ is approximate for small $\sigma n^{-\frac{1}{2}}$. If $\sigma n^{-\frac{1}{2}}$ is small, $(\partial^k/\partial x^k)[\hat{s}(x)-s(x)]$ for $k=0$, 1, 2, and $\hat{x}-x_0$ are of the order of $\sigma n^{-\frac{1}{2}}$. Since $\hat{s}$ is a maximum at $\hat{x}$ and s is a maximum at x_0, by Taylor series expansion about x_0 we have

$$\frac{\partial}{\partial x}\hat{s}(\hat{x})=0$$
$$=\frac{\partial}{\partial x}e(x_0)+\frac{\partial^2}{\partial x^2}s(x_0)(\hat{x}-x_0)+O(\sigma^2n^{-1}), \tag{15}$$

where $(\partial^2/\partial x^2)s(x_0)<0$. Consequently, the small $\sigma n^{-\frac{1}{2}}$ rms error of $\hat{x}$ is

$$\mathrm{rms}(\hat{x}-x_0)=\left[-\frac{\partial^2}{\partial x^2}s(x_0)\right]^{-1}\left\{E\left[\frac{\partial}{\partial x}e(x_0)\right]^2\right\}^{\frac{1}{2}}. \tag{16}$$

For the situation where $\Sigma=I$ and $\Phi'\Phi\approx(1/n)I$,

$$\mathrm{rms}(\hat{x}-x_0)=\sigma n^{-\frac{1}{2}}\left[-\frac{\partial^2}{\partial x^2}s(x_0)\right]^{-1}\times\left\{\sum_{m=1}^{M}\gamma_m^2\left[\frac{\partial}{\partial x}\varphi_m(x_0)\right]^2\right\}^{\frac{1}{2}}. \tag{17}$$

Given $\hat{x}$, we will now use the method of scoring to iterate to the maximum-likelihood estimator. The method of scoring is just the Newton–Raphson procedure for solving the equation $(\partial/\partial x)L(\mathbf{y}|x)=0$ modified by replacing $-(\partial^2/\partial x^2)\log L(\mathbf{y}|x)$ by its expected value, the Fisher information $I(x)$, evaluated at $x=\hat{x}$,[7] i.e., the next iterate of the source depth is

$$\hat{x}_0=\hat{x}+\frac{1}{I(\hat{x})}\frac{\partial}{\partial x}\log L(\mathbf{y}|\hat{x}), \tag{18}$$

where

$$I(x)=-E\frac{\partial^2}{\partial x^2}\log L(\mathbf{y}|x). \tag{19}$$

From Eq. 8 it is clear that

$$\frac{\partial}{\partial x}\log L(\mathbf{y}|x)=\boldsymbol{\varphi}_x'(x)\mathbf{D}\boldsymbol{\Phi}'\boldsymbol{\Sigma}^{-1}[\mathbf{y}-\boldsymbol{\Phi}\mathbf{D}\boldsymbol{\varphi}(x)], \tag{20}$$

where D is the diagonal matrix whose mth element is $d_m=\gamma_m^{-1}a_m r^{-\frac{1}{2}}\cos\kappa_m r$. Applying Eq. 20 to Eq. 18, the "scored" estimate of depth is

$$\hat{x}_0=\hat{x}+I^{-1}(\hat{x})\boldsymbol{\varphi}_x'(\hat{x})\mathbf{D}\boldsymbol{\Phi}'\boldsymbol{\Sigma}^{-1}[\mathbf{y}-\boldsymbol{\Phi}\mathbf{D}\boldsymbol{\varphi}(\hat{x})], \tag{21}$$

where

$$I(x)=\boldsymbol{\varphi}_x'\mathbf{D}\boldsymbol{\Phi}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Phi}\mathbf{D}\boldsymbol{\varphi}_x. \tag{22}$$

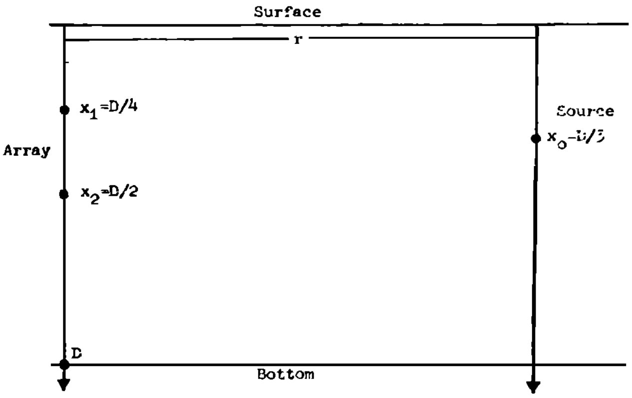

FIG. 2. Two-element vertical array.

The maximum-likelihood estimate of x_0 is found by repeating the scoring procedure until it eventually converges.[11] For small $\sigma n^{-\frac{1}{2}}$, the initial estimate given by $\hat{x}$ will be close to the true x_0, and thus the scoring procedure should converge quickly. For notational simplicity let $\hat{x}_0$ denote the final iterate—the MLE.

The rms error of the maximum-likelihood estimator is

$$\text{rms}(\hat{x}_0 - x_0) = I^{-\frac{1}{2}}(x_0). \tag{23}$$

For a fixed σ and large n, the MLE has the smallest error of any estimator of x_0 which is a smooth function of x_0.[7] However, for small n, the initial estimator $\hat{x}$ can have a smaller error than $\hat{x}_0$ provided σ is small. The optimal properties of the MLE hold in the limit as $n \rightarrow \infty$, rather than as $\sigma \rightarrow 0$. In the next section, a simple example is given which illustrates the processing technique and the error analysis of the simple estimator $\hat{x}$ and the MLE $\hat{x}_0$.

IV. A SIMPLE EXAMPLE

For the homogeneous ocean waveguide discussed in the Introduction, the lower mode eigenfunctions are of the form $\varphi_m(x) = \sqrt{2}\sin m\pi(x/D)$. Suppose that M/D is small. Then $\gamma_m = m\pi/D$ is small for $m \leq M$, and thus $\kappa_m \approx \omega/c$. If the distance between source and array is $r = c/\omega$, then $\cos\kappa_m r \approx 1$ for $m \leq M$.

In this section the statistical analysis of vertical steering is illustrated for a simple source model and a two sensor array. Suppose that a source at depth $x_0 = D/3$ excites only the two lowest modes. More specifically, let

$$s(x) = \varphi_1(x) + \tfrac{1}{2}\varphi_2(x). \tag{24}$$

Note that $s(x)$ has its maximum for $0 \leq x \leq D$ at $x = D/3$ (Fig. 2). The attenuated exitations are $a_1 = 2(r/6)^{\frac{1}{2}}$ and $a_2 = (r/6)^{\frac{1}{2}}$. Moreover, $(\partial/\partial x)\varphi_1(x_0) = (\sqrt{2}/2)(\pi/D)$, $(\partial/\partial x)\varphi_2(x_0) = -\sqrt{2}(\pi/D)$, and

$$(\partial^2/\partial x^2)s(x_0) = -(3\sqrt{6}/2)(\pi/D)^2.$$

Let x_1 and x_2 denote the depths of the two sensors. Then

$$\Phi = \sqrt{2}\begin{pmatrix} \sin\pi(x_1/D) & \sin 2\pi(x_1/D) \\ \sin\pi(x_2/D) & \sin 2\pi(x_2/D) \end{pmatrix}.$$

Since we are dealing with an example, let us further simplify the algebra by setting $x_1 = D/4$ and $x_2 = D/2$. Then

$$\Phi = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$$

and thus

$$(\Phi'\Phi)^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{pmatrix}.$$

Clearly rms $(\hat{x} - x_0)$ does not depend on D for any choice of x_1 and x_2. Consequently, from Eqs. (14) and (16), the small σ rms error of $\hat{x}$ is

$$\text{rms}\left(\hat{x} - \frac{D}{3}\right) = (0.57)^{\frac{1}{2}}\sigma, \tag{25}$$

assuming that the ambient noise at x_1 and x_2 are uncorrelated. It is easy to show that

$$\text{rms}\left(\hat{x}_0 - \frac{D}{3}\right) = (1.44)^{\frac{1}{2}}\sigma. \tag{26}$$

Therefore, when σ is small and initial estimator $\hat{x}$ is more precise than the maximum-likelihood estimator in this two-sensor example.

Different sensor positions would give a different $(\Phi'\Phi)^{-1}$ matrix in Eq. (14). For example, if the aperture $x_1 - x_2$ is small, the eigenvalues of $(\Phi'\Phi)^{-1}$ are of the order $(x_1 - x_2)^{-2}$, and thus σ must be of the order $(x_1 - x_2)^{-1}$ in order to insure the validity of the approximation for the rms $(\hat{x} - x_0)$. For a given σ the error in $\hat{x}$ is reduced by using $n > 2$ sensors in the array. However, the rms of $\hat{x}$ and $\hat{x}_0$ are not in general proportional to $n^{-\frac{1}{2}}$, but rather are inversely proportional to the average eigenvalue of $\Phi'\Sigma^{-1}\Phi$.

V. CONCLUSION

It is possible to estimate the depth of a distant stationary source in a waveguide by the appropriate "steering" of a vertical array. Rather than employing delay-and-sum beam forming, the optimal signal processor depends on the eigenfunctions of the guide. The estimator of the mode amplitudes is a linear function of the filtered output from the sensors, but the depth estimator is nonlinear.

The error in the depth estimate depends on the power and coherence of the ambient noise in the source's frequency band, the aperture and geometry of the array, and the observation period. If the source and medium are sufficiently stationary and the array is deep enough in the guide, an accurate estimate of source depth can be obtained. Given the eigenfunctions, the processing technique can be applied to any waveguide in which a source is detected by an array of sensors.

*This work was supported by the Office of Naval Research under contract.

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