Error analysis of velocity and direction measurements of plane waves using thick large-aperture arrays

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The statistical properties of estimators for plane-wave parameters are discussed. A thick large-apetture array is assumed to be detecting a signal emanating from an unknown continuous wave source. The signal is assumed to be a plane wave embedded in a spatially incoherent Gaussian noise field. Maximum-likelihood estimators of velocities and directions are derived and their root-mean-square errors are obtained, for single and multiple arrivals for linear arrays and square arrays with a large number of sensors. The rms errors of the propagation parameter estimators are proportional to K^{-1} for a square-lattice array of K sensors and to $K^{-3/2}$ for a linear array of K sensors. The rms error of the waveform estimator, however, is proportional to $K^{-1/2}$ in each case. Thus, for the propagation parameters, the figure of merit for rms signal-to-noise gain of the array is K for a square array of K sensors and $K^{3/2}$ for a linear array of K sensors.

Subject Classification: 15.2; 15.3.

INTRODUCTION

Consider a large aperture linear or planar array immersed in a waveguide containing a set of farfield sources of interest. In the region of the array, the signals from these sources are approximately plane waves.[1] Given a finite record of simultaneously observed sensor outputs, we wish to estimate the directions and/or the phase and group velocities of the plane waves produced by the sources. Assume that the noise field is stationary in space and time, Gaussian, and incoherent across the array.

For a given mode of a single source, delay-and-sum filtering (beam forming) is the method widely used to estimate the waveform of the signal, provided that the waveguide is essentially nondispersive in the region about the array. The direction and velocity are estimated by searching over a range of possible directions and velocities for the parameter values which maximize the output energy of delay-and-sum filtering; i.e., delays for each sensor channel are computed for an assumed direction and velocity, the channels are delayed and summed, and the sum is squared and integrated over the duration of the observation period. For the situation where the velocity is known, the search method is equivalent to steering the array in the direction which maximizes the energy in the main beam. Ignoring finite record end effects, Levin[2] sketches a proof that this nonlinear search provides least-squares estimators of the signal parameters and approximates their variances for arrays with a large number of sensors and a high signal-to-noise ratio. However, if the waveguide is dispersive and the signal has a wide bandwidth, the group and phase velocities depend on frequency. This produces a systematic distortion of the signal waveform from sensor to sensor and causes a reduction of signal enhancement for the steered array. Clay and Hinich[3] have developed and analyzed an *ad hoc* procedure for estimating the phase velocities and direction for the dispersive situation.

Even if the waveguide is nondispersive, the presence of multiple sources requires the modification of simple delay-and-sum beam forming.[4] Levin's method can be employed in a sequential manner for estimating the parameters of overlapping plane waves from multiple sources. For purposes of exposition, suppose there are two waves of known velocity propagating across the array from two sources at unknown locations. The array is first steered in the direction of a source which gives maximum energy in the beam. Its estimated waveform is appropriately delayed and then is subtracted from each channel. The delay-and-sum search procedure is reemployed to estimate the direction of the weaker source. Anderson and Rudnick[5] have derived the array processing gain for this adaptive procedure, assuming stochastic signals. The statistical properties of the bearing estimates are unknown, but from preliminary analysis using artificial data we have found that the estimated direction of the weaker source can have a considerable bias, even with large signal-to-noise ratios, if the search is not made over a closely spaced grid of possible bearings, or if the true velocities are somewhat different from the ones assumed for the beam forming. The number of bearings and velocities which should be searched is not specified by Levin.

For a single signal which is modeled as a section of a stationary nondispersive propagating stochastic field embedded in a noise field of known properties, the optimum array processing has been analyzed by several

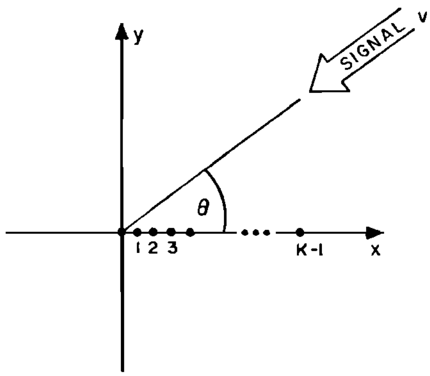

FIG. 1. Incoming signal at a linear array. The signal has a velocity of propagation v and has a direction θ.

investigators.[1,6–14] Large array analysis of a split-beam linear array processor is given by MacDonald and Schultheiss.[15] When the noise covariance is unknown, optimal sonar processing is discussed by Liggett.[16]

This paper develops maximum-likelihood estimators for the *directions* and *velocities* of a sum of plane-wave signals of unknown waveforms which are propagating across an aperture array consisting of a large number of sensors in the presence of additive Gaussian noise.[17]

The estimators are derived from the frequency–wavenumber ($\omega-\kappa$) spectrum which is computed from a sample of digital records of each sensor channel. For the case of a single nondispersive monochromatic plane wave, the estimated wavelength, velocity, and direction correspond to the frequency and wavenumber of the point of maximum energy of the $\omega-\kappa$ spectrum. In our method the maximum number of grid points to be searched is specified, the statistical properties are derived, and the multiple-source problem is treated. Assuming that the array contains a large number of sensors and that the waves are coherent across the array, we shall show that the precision of the velocity and bearing estimators is considerably greater than the precision of the waveform estimators. To be precise, the root-mean-square errors of the propagation parameter estimators are proportional to K^{-1} for a square lattice array of K sensors and to $K^{-\frac{3}{2}}$ for a linear array of K sensors; whereas the rms error of the waveform estimator is proportional to $K^{-\frac{1}{2}}$ in each case. The figure of merit for rms signal-to-noise gain of an array is generally given as $K^{\frac{1}{2}}$, but if one is interested in the propagation parameters the figure of merit should be K for a square lattice array of K sensors and $K^{\frac{3}{2}}$ for a linear array of K sensors.

I. PLANE WAVES PROPAGATING ACROSS A LINEAR ARRAY

The maximum-likelihood processing is based upon finding maxima in wavenumber spectra. Before developing the maximum-likelihood method, let us define the signal and array.

Let us consider the simplest case, that of a horizontal linear array of K *equally spaced* sensors which is detecting a horizontally propagating plane wave, Fig. 1. Assume that the plane wave, at time t and position x on the array axis, is defined by

$$s(t,x)=\frac{1}{\omega_b-\omega_a}\int_{\omega_a}^{\omega_b}A(\omega)\times\exp\left\{-i\left[\omega\left(t-\frac{x\cos\theta}{v(\omega)}\right)-\varphi(\omega)\right]\right\}d\omega, \quad (1)$$

where $A(\omega)$ and $\varphi(\omega)$ are, respectively, the amplitude and phase of the wave components, $\omega_b-\omega_a$ is the bandwidth, $\theta(0\leq\theta\leq\pi)$ is the direction of propagation with respect to the x axis, and $v(\omega)$ is the phase velocity. For each ω,

$$\alpha(\omega)=\omega\cos\theta/v(\omega) \quad (2)$$

is the x component of the wavenumber vector of the propagating wave. The actual dependence of α upon ω can be quite complicated and is usually obtained numerically for a particular waveguide. Over a narrow band (ω_a,ω_b) one can make an analytical approximation to the numerical relationship.[18] We include the possibility of multiple values of α for a given ω by writing the signal as

$$s(t,x)=\sum_{m=1}^{M}\frac{1}{\omega_b-\omega_a}\int_{\omega_a}^{\omega_b}A_m(\omega)\times\exp\left\{-i\left[\omega\left(t-\frac{x\cos\theta_m}{v_m(\omega)}\right)-\varphi_m(\omega)\right]\right\}d\omega. \quad (3)$$

Let x_k be the coordinate of the kth sensor on the array axis and let $p(t,x_k)$, $t=0, \tau, \cdots, (T-1)\tau$, be the sampled output of the kth sensor, $k=0, 1, \cdots, K-1$. To avoid aliasing let $\tau\leq\pi/\omega_b$. Correspondingly, one also has a spatial sampling theorem. We will assume the separation d between sensors is less than half of the shortest phase wavelength, i.e., $d\leq\frac{1}{2}\lambda_b$ or π/α for ω in the band.

To simplify notation we choose the time and space units such that $\tau=1$ and $d=1$. Thus the wavenumber is measured in units of d^{-1}. Let the signal at the kth sensor be

$$p(t,x_k)=s(t,x_k)+n(t,x_k),$$

where $n(t,x_k)$ is spatially incoherent Gaussian noise with total power σ^2 in (ω_a,ω_b), i.e.,

$$E[n(t,x_k)]=0,$$

$$E[n(t,x_k)n(t',x_{k'})]=0 \quad \text{for } k\neq k',$$

$$\sigma^2=\frac{1}{2\pi}\int_{\omega_a}^{\omega_b}S_n(\omega)d\omega,$$

where $S_n(\omega)$ is the power spectrum of the noise.

The discrete Fourier transformation of each sensor channel output observed at $t=0, 1, \cdots, T-1$ is computed at the B frequencies $\omega_j=2\pi j/T$, where j is an integer and $\omega_a \leq \omega_j \leq \omega_b$. The Fourier transform pair is

$$p(t,x_k)=\frac{1}{B}\sum_{j=0}^{T-1} P(\omega_j,x_k)e^{-i\omega_j t}, \quad t=0, 1, \cdots, T-1,$$
$$P(\omega_j,x_k)=\frac{B}{T}\sum_{t=0}^{T-1} p(t,x_k)e^{i\omega_j t}, \quad j=0, 1, \cdots, T-1. \tag{4}$$

We can view B as the number of narrow bands of bandwidth $\Delta\omega=2\pi/T$ in (ω_a,ω_b) and thus $B=(\omega_b-\omega_a)/\Delta\omega$. If the observation period T is greater than the time taken by the wave group to traverse the array, and if the changes of $A(\omega)$ and $\varphi(\omega)$ are small for $\Delta\omega=2\pi/T$, they can be replaced by their mean values. From Eqs. 1, 2, and 4 we can write

$$P(\omega_j,x_k)=A \exp\{i(\alpha x_k+\varphi)\}+N(\omega_j,x_k), \tag{5}$$

where α, A, and φ depend on ω_j. For each ω_j in the band (ω_a,ω_b) and x_k

$$N(\omega_j,x_k)=\frac{B}{T}\sum_{t=0}^{T-1} n(t,x_k)e^{i\omega_j t} \tag{6}$$

is a complex Gaussian random variable with

$$E[N(\omega_j,x_k)]=0, \quad E[|N(\omega_j,x_k)|^2]=B\sigma^2, \tag{7}$$

assuming that the noise spectrum is flat in the (ω_a,ω_b) band. Moreover, for each k, $N(\omega_j,x_k)$ and $N(\omega_l,x_k)$ are approximately independent random variables for $\omega_j\neq\omega_l$, both in the (ω_a,ω_b) band.

II. MAXIMUM-LIKELIHOOD ESTIMATION FOR SINGLE ARRIVALS

For a given ω, consider the following estimate of the ω,α (power) spectrum:

$$S_K(\omega,\alpha)=\frac{1}{K}\left|\sum_{k=0}^{K-1} P(\omega,x_k)e^{-i\alpha x_k}\right|^2. \tag{8}$$

As indicated in Hinich and Clay[19] and discussed in more mathematical detail in Walker,[20] for $K\rightarrow\infty$

$$\begin{aligned} K^{-1}S_K(\omega,\alpha)&=A^2(\omega)+O(K^{-\frac{1}{2}}), &&\text{if} \quad \alpha=\alpha(\omega),\\ &=O(K^{-\frac{1}{2}}), &&\text{if} \quad \alpha\neq\alpha(\omega), \end{aligned} \tag{9}$$

where $O(K^{-\frac{1}{2}})$ denotes a variable with second moments of order K^{-1}. Consequently $S_K(\omega,\alpha)$ has a pronounced maximum at $\alpha=\alpha(\omega)$ for large K, which suggests the following estimator of $\alpha(\omega)$. Let $\hat{\alpha}$ be the value of α which maximizes $S_K(\omega,\alpha)$ for the given ω, i.e.,

$$S_K(\omega,\hat{\alpha})=\max_{-\pi\leq\alpha\leq\pi} S_K(\omega,\alpha). \tag{10}$$

From the work of Walker[20] and Whittle,[21] Hinich and Shaman[22] show that $\hat{\alpha}$ is the maximum-likelihood estimator of α and for large K its distribution is approximately normal (Gaussian). The mean square error (mse) of $\hat{\alpha}$ is

$$E(\hat{\alpha}-\alpha)^2=6B/K^3\rho, \tag{11}$$

where $\rho=(A/\sigma)^2$ is the signal-to-noise (S/N) ratio.

The spatial resolution of the array is of the order $2\pi/K$. The range of α is from $-\pi$ to π, and the above estimation procedure requires estimation at increments of α which should be spaced at least as close as $2\pi/K$. Suppose we compute $S_K(\omega,\alpha)$ at the K^2+1 equally spaced wave numbers (assuming for simplicity that K^2 is even)

$$\alpha_p=2\pi pK^{-2}, \quad p=0, \pm1, \cdots, \pm\tfrac{1}{2}K^2. \tag{12}$$

Using the Fast Fourier Transform to compute

$$\sum_{k=0}^{K-1} P(\omega,x_k)e^{-i\alpha_p x_k}$$

from a sequence of $P(\omega,x_k)$ augmented by zeroes, we can quickly and efficiently compute the $S_K(\omega,\alpha_p)$ for large values of K^2. Now let $\alpha_{\hat{p}}$ be the discrete wave number which maximizes $S_K(\omega,\alpha_p)$, i.e., $\alpha_{\hat{p}}=2\pi\hat{p}/K^2$ for some integer $\hat{p}$ such that

$$S_K(\omega,\alpha_{\hat{p}})=\max_{\alpha_p} S_K(\omega,\alpha_p). \tag{13}$$

Clearly

$$\alpha_{\hat{p}}-\hat{\alpha}=O(K^{-2}), \tag{14}$$

which is less than $O(K^{-\frac{3}{2}})$, the order of the square-root mse of $\hat{\alpha}$.

Formula 11 is valid even if the sensors are not equally spaced, provided that the resolution of the array is of the order $2\pi/K$ and the side lobes are small, i.e., there are a sufficient number of closely spaced sensors in an array of length K. This robustness to relatively minor perturbations of the array geometry follows from a straightforward generalization of the proofs in Hinich and Shaman.[22]

Now suppose that we have computed $\hat{\alpha}(\omega_j)$ at each $\omega_j=2\pi j/T$ in the band (ω_a,ω_b). Let us present the maximum-likelihood estimator of the direction θ or phase velocity $v(\omega)$. Since the array is linear, it is not possible to obtain proper estimates of both θ and $v(\omega)$. In Sec. IV we will develop maximum-likelihood estimators of θ and $v(\omega)$ using the output of a planar array.

First let us assume that $v(\omega)$ is known for $\omega_a\leq\omega\leq\omega_b$. The maximum-likelihood estimator for θ based on the ωth component of the array output is

$$\hat{\theta}(\omega)=\arccos\{[v(\omega)/\omega]\hat{\alpha}(\omega)\}. \tag{15}$$

The principal value of $\arccos(x)$ ranges over an interval of length π, and we have restricted θ to $[0,\pi]$.

In order to obtain a simple expression for $\hat{\theta}$, the maximum-likelihood estimator of θ based upon the $\hat{\theta}(\omega)$, $\omega_a\leq\omega\leq\omega_b$, let us assume that the signal is white, i.e., $A(\omega)=A$ in the band. The estimator $\hat{\theta}$ is a weighted

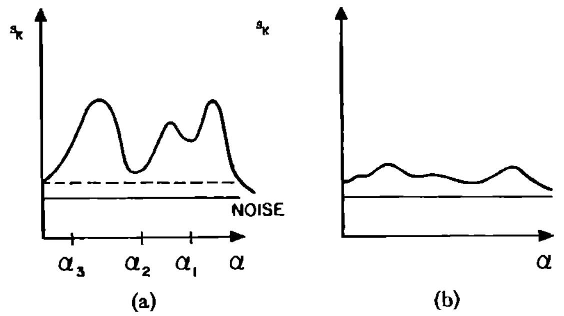

FIG. 2. α spectrum $S_K(\alpha,\omega)$. (a) High signal-to-noise power ratio. The possible multiple values of α, α_m are shown. (b) Low signal-to-noise power ratio.

average of the $\hat{\theta}(\omega)$, where the weights depend on the asymptotic (large K) mse of $\hat{\theta}(\omega)$.

Applying Eqs. 11 and 15 with $\arccos(x)$ expanded in a Taylor series about $x=v\alpha/\omega$, we find that the asymptotic mse of $\hat{\theta}(\omega)$ is

$$E[\hat{\theta}(\omega)-\theta]^2=\frac{3B\lambda^2(\omega)}{2K^3\rho(\pi\sin\theta)^2};\quad \theta\neq 0,\pi, \tag{16}$$

where $\lambda(\omega)=2\pi v(\omega)/\omega$ is the wavelength of the ωth frequency component.

The maximum-likelihood estimator of θ is then

$$\hat{\theta}=\frac{1}{\omega_b-\omega_a}\int_{\omega_a}^{\omega_b}\frac{\bar{\lambda}}{\lambda(\omega)}\hat{\theta}(\omega)d\omega, \tag{17}$$

where

$$\bar{\lambda}=\left[\frac{1}{\omega_b-\omega_a}\int_{\omega_a}^{\omega_b}\lambda^{-1}(\omega)d\omega\right]^{-1}$$

and $d\omega=2\pi/T$. From Eq. 16 the asymptotic mse of $\hat{\theta}$ is

$$E(\hat{\theta}-\theta)^2=3\bar{\lambda}^2/2K^3\rho(\pi\sin\theta)^2 \tag{18}$$

when $\theta\neq 0,\pi$ and is of order K^{-3} due to the bias when $\theta=0$ or π. This is the asymptotic mse of the estimator based on phase differences which was developed by Clay and Hinich[3] and of the estimator given by Capon and Green.[10] Both estimators assume a high S/N ratio for a fixed number of sensors. The fact that the same mse is obtained by assuming a fixed S/N ratio and many sensors is a nonobvious result.

Now let us assume that the direction is known ($\theta\neq\frac{1}{2}\pi$) and the phase velocity is to be estimated. For a given ω, the maximum-likelihood estimator of $v(\omega)$ is

$$\hat{v}(\omega)=\omega\cos\theta/\hat{\alpha}(\omega). \tag{19}$$

Once again, by employing a Taylor-series approximation method for a white signal and Eq. 11, we find that the asymptotic proportional mse of $\hat{v}(\omega)$ is

$$E\left[\frac{\hat{v}(\omega)}{v(\omega)}-1\right]^2=\frac{3B\lambda^2(\omega)}{2K^3\rho(\pi\cos\theta)^2}. \tag{20}$$

If the wave is *nondispersive*, the phase velocity is the same for all frequencies. In this case the maximum-likelihood estimator of v is

$$\hat{v}=\frac{1}{\omega_b-\omega_a}\int_{\omega_a}^{\omega_b}\frac{\omega}{\bar{\omega}}\hat{v}(\omega)d\omega, \tag{21}$$

where $\bar{\omega}=(\omega_a+\omega_b)/2$. From Eq. 20 the asymptotic proportional mse of $\hat{v}$ is $3\bar{\lambda}^2/[2K^3\rho(\pi\cos\theta)^2]$.

The above results can be neatly summarized for the *monochromatic wave* sampled at the Nyquist interval, i.e., where $\omega_b=\omega_a=\omega_0$, $\lambda_0=2\pi v/\omega_0$, and $d=\frac{1}{2}\lambda_0$. Setting $\omega=\omega_0$ in Eqs. 15, 16 and Eqs. 19, 20, we find the (square) rms errors are

$$\begin{aligned}\text{rmse}(\hat{\theta})&=\left(\frac{6}{\rho}\right)^{\frac{1}{2}}\frac{1}{K^{\frac{3}{2}}\pi\sin\theta},\quad \theta\neq 0,\pi\\ \text{rmse}\frac{\hat{v}}{v}&=\left(\frac{6}{\rho}\right)^{\frac{1}{2}}\frac{1}{K^{\frac{3}{2}}\pi|\cos\theta|},\quad \theta\neq\tfrac{1}{2}\pi.\end{aligned} \tag{22}$$

Thus the rms errors of the estimators are inversely proportional to the square root of the S/N ratio times the number of sensors raised to the power $\frac{3}{2}$. Note that the precision of the direction estimate is highest when the array is broadside to the propagation, whereas the velocity estimate is most precise for the endfire situation.

III. MAXIMUM-LIKELIHOOD PROCESSING FOR MULTIPLE ARRIVALS

Multiple arrivals at an array can be observed because (i) there are several sources, (ii) several travel paths or modes, or (iii) a combination of (i) and (ii). In a single-path situation sources having about the same frequency spectra have α components along the array which depend upon the direction to the source. Again, wave-number spectrum analysis would determine the directions of the sources. The multiple arrivals from a single source have different components of α as shown in Fig. 2. For a known medium, theoretical values of α can be used in identifying the source. In most acoustical problems we expect several sources and each source can have several arrivals.

The Gaussian noise $n(t,x_k)$ has the properties assumed in the previous section. For sake of exposition we will only deal with two special but important forms for $s(t,x_k)$: the case of multiple farfield monochromatic sources which generate the same mode of propagation at the array location during the observation time, i.e., $v_1=v_2=\cdots=v_M$, and the case of a single source generating multiple modes propagating in the same direction, i.e., $\theta_1=\theta_2=\cdots=\theta_M$.

First let $v_1=\cdots=v_M=v$ and assume that the source directions θ_m are distinct and satisfy $0<\theta_m<\pi$, $m=1,\cdots,M$. In the case of multiple sources it is necessary to have some method of matching the wave-number estimates with the different sources according to their energies. For this purpose we have to consider the amplitudes A_m, $m=1, \cdots, M$. Without loss of generality we assume $A_1^2 > \cdots > A_M^2$. The wavenumber for the mth plane wave in the signal (Eq. 3) is

$$\alpha_m(\omega_0) = 2\pi \cos\theta_m/\lambda_0. \tag{23}$$

As before, compute $S_K(\omega_0,\alpha)$ from the $P(\omega_0,x_k)$ for a fine grid of wavenumbers in $(-\pi,\pi)$. Let $\hat{\alpha}_1$ be the wavenumber which maximizes $S_K(\omega_0,\alpha)$. Excluding $\hat{\alpha}_1$, let $\hat{\alpha}_2$ be the wavenumber which maximizes $S_K(\omega_0,\alpha)$. In general, let $\hat{\alpha}_m$ be the wavenumber excluding the set $\{\hat{\alpha}_1,\cdots,\hat{\alpha}_{m-1}\}$ which maximizes $S_K(\omega_0,\alpha)$, i.e., $\hat{\alpha}_m(\omega)$ is the wavenumber which is associated with the mth *greatest maximum* of $S_K(\omega_0,\alpha)$, where we have searched over a specified grid of wavenumbers. Then the maximum-likelihood estimator of A_m is

$$\hat{A}_m = \frac{1}{K}\left|\sum_{k=0}^{K-1} P(\omega_0,x_k)e^{-i\hat{\alpha}_m x_k}\right|, \quad m=1,\cdots,M. \tag{24}$$

That is, the mth greatest maximum of $S_K(\omega_0,\alpha)$ corresponds to the source which has mth greatest estimated energy. The mse of $\hat{\alpha}_m$ is

$$E[\hat{\alpha}_m - \alpha_m]^2 = 6/K^3\rho_m d^2. \tag{25}$$

Moreover $\hat{\alpha}_m$ and $\hat{\alpha}_n$ are approximately independent for large K for $m \neq n$.

In order to discuss the resolving ability of the array, let us consider two $(M=2)$ closely spaced monochromatic sources which are located approximately broadside to the array, i.e., $\theta_2 - \theta_1 = \Delta\theta$ is small and $\theta_1 \approx \frac{1}{2}\pi$. Further, suppose the second source is considerably stronger than the first, and consequently $\rho_2 \gg \rho_1$. In order to resolve the weaker source the rms error of $\hat{\theta}_1$ must be smaller than $\Delta\theta$. If the medium is nondispersive and $d=\frac{1}{2}\lambda_0$, it follows from Eq. 22 that the weaker source is resolved if

$$K > (6/\rho_1)^{\frac{1}{3}}(1/\pi\Delta\theta)^{\frac{2}{3}}. \tag{26}$$

If the sources are at 30° azimuth $(\theta_1=\pi/6)$, then the sufficient lower bound is $2^{\frac{2}{3}}$ times the right-hand side of Eq. 26.

Now assume that $\theta_1 = \cdots = \theta_M = \theta$ and that the phase velocities v_m are distinct. Assuming θ is known, we see that the maximum-likelihood estimator of v_m is as in Eq. 19, and its proportional mse is given by Eq. 20 with λ and ρ subscripted by m.

Given a single monochromatic source, consider the problem of resolving the two lowest modes when it is assumed that the difference Δv between the two corresponding v_m values is small. If the source lies along the axis of the array and $\rho_1=\rho_2=\rho$, then the modes can be resolved if

$$K > (6/\rho)^{\frac{1}{3}}(v/\pi\Delta v)^{\frac{2}{3}},$$

where $d=\pi v/\omega_0=\frac{1}{2}\lambda_0$.

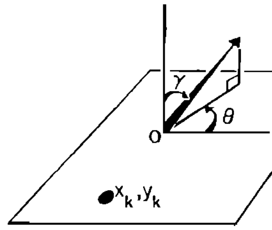

FIG. 3. kth sensor at $\mathbf{x}_k=(x_k,y_k,0)'$. Array plane, direction of propagation (arrow), elevation angle of propagation γ, and azimuth angle of propagation in the plane θ are shown.

IV. ESTIMATION FOR SQUARE-LATTICE ARRAYS

In this section we treat estimation for a square-lattice array of equally spaced sensors. We will discuss the case of single arrivals, since the multiple-arrival analysis is analogous to that given in Sec. III.

Let us assume that a collection of $K=n^2$ sensors is arranged in a square-lattice array with spacing $d=1$ between each pair of adjacent sensors along the axes of the square. The array is located on a plane in the three-dimensional Euclidean space R^3. Let us choose the coordinate system of R^3 so that the third axis is perpendicular to the array plane. The coordinates of the kth sensor in the array are thus $\mathbf{x}_k=(x_k,y_k,0)'$. The x and y components of the wavenumber vector are

$$\alpha(\omega) = \frac{\omega \sin\gamma \cos\theta}{v(\omega)}, \quad \beta(\omega) = \frac{\omega \sin\gamma \sin\theta}{v(\omega)}, \tag{27}$$

where $\gamma(0<\gamma<\pi, \gamma\neq\frac{1}{2}\pi)$ is the angle of propagation with respect to the normal to the array plane, $\theta(0<\theta<2\pi, \theta\neq\frac{1}{2}\pi, \pi, \frac{3}{2}\pi)$ is the azimuth angle of propagation in the plane, and $v(\omega)$ is the phase velocity (Fig. 3).

We proceed as in the one-dimensional case. The output of each sensor is sampled at the times $t=0, \tau, \cdots, (T-1)\tau$, with $\tau \leq \pi/\omega_b$, and for notational convenience we choose the time unit so that $\tau=1$. The output of the kth sensor is $p(t,\mathbf{x}_k)=s(t,\mathbf{x}_k)+n(t,\mathbf{x}_k)$, where $n(t,\mathbf{x}_k)$ is again spatially incoherent Gaussian noise with total power σ^2 in (ω_a,ω_b). Set

$$S_K(\omega,\alpha,\beta) = \frac{1}{K}\left|\sum_{k=0}^{K-1} P(\omega,\mathbf{x}_k)e^{-i(\alpha x_k+\beta y_k)}\right|^2,$$

where $P(\omega,\mathbf{x}_k)$ is the Fourier transform of the kth sensor output. We define $(\hat{\alpha},\hat{\beta})$ by

$$S_K(\omega,\hat{\alpha},\hat{\beta}) = \max_{-\pi\leq\alpha,\beta\leq\pi} S_K(\omega,\alpha,\beta). \tag{28}$$

Hinich and Shaman[22] show that $(\hat{\alpha},\hat{\beta})$ is the maximum-likelihood estimator of (α,β) and that its distribution is approximately bivariate normal (Gaussian) for large K.

The maximum-likelihood estimator of the azimuth θ based on the ωth component of the array output is $\arctan(\hat{\beta}/\hat{\alpha})$ for $\hat{\beta}>0$, and $\pi+\arctan(\hat{\beta}/\hat{\alpha})$ for $\hat{\beta}<0$. The large K mse of $\hat{\theta}$ is

$$E(\hat{\theta}-\theta)^2 = 3\bar{\lambda}^2/2K^2\rho(\pi\sin\gamma)^2. \qquad (29)$$

Let us assume that $v(\omega)$ is known for $\omega_a \leq \omega \leq \omega_b$. Then the maximum-likelihood estimator of the elevation angle γ based on the ωth component of the array output is

$$\hat{\gamma}(\omega) = \arcsin\{[v(\omega)/\omega][\hat{\alpha}^2(\omega)+\hat{\beta}^2(\omega)]^{\frac{1}{2}}\}$$

with $0 \leq \hat{\gamma}(\omega) \leq \frac{1}{2}\pi$. However, $0<\gamma<\pi$ with $\gamma \neq \frac{1}{2}\pi$. In other words, using this estimation procedure we cannot determine whether the direction of propagation of the wave is up or down. This ambiguity results because a two-dimensional array is being used to analyze a three-dimensional problem. The ambiguity is easily resolved by determining the direction of flow of energy using either sensor information or *a priori* knowledge of the location of the source. If we were to measure γ with respect to the plane of the array, so that $-\frac{1}{2}\pi<\gamma<\frac{1}{2}\pi$ with $\gamma \neq 0$, then the difficulty would occur in the estimation of θ. Specifically, using this estimation procedure we would be unable to determine whether the wave is moving to the left or the right, say. The large K mse of $\hat{\gamma}$ is

$$E(\hat{\gamma}-\gamma)^2 = 3\bar{\lambda}^2/2K^2\rho(\pi\cos\gamma)^2. \qquad (30)$$

The phase velocity $v(\omega)$ can be estimated if the elevation angle γ is known. For fixed ω the maximum-likelihood estimator of $v(\omega)$ is

$$\hat{v}(\omega) = \omega\sin\gamma/[\hat{\alpha}^2(\omega)+\hat{\beta}^2(\omega)]^{\frac{1}{2}},$$

and use of the Taylor-series method again gives for the large K proportional mse

$$E[\hat{v}(\omega)/v(\omega)-1]^2 = 3B\lambda^2(\omega)/2K^2\rho(\pi\sin\gamma)^2 \qquad (31)$$

for a white signal.

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