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Signal Processing 85 (2005) 1557–1562

SIGNAL PROCESSING

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Detecting finite bandwidth periodic signals in stationary noise using the signal coherence spectrum

Melvin J. Hinich[a,*], Phillip Wild[b]

[a]Applied Research Laboratories, University of Texas at Austin, P.O. Box 8029, Austin, TX 78713-8029, USA
[b]Centre for Economic Policy Modeling, School of Economics, University of Queensland, St Lucia, QLD, 4072, Australia

Received 14 June 2004

Abstract

All signals that appear to be periodic have some sort of variability from period to period regardless of how stable they appear to be in a data plot. A true sinusoidal time series is a deterministic function of time that never changes and thus has zero bandwidth around the sinusoid's frequency. A zero bandwidth is impossible in nature since all signals have some intrinsic variability over time. Deterministic sinusoids are used to model cycles as a mathematical convenience. Hinich [IEEE J. Oceanic Eng. 25 (2) (2000) 256–261] introduced a parametric statistical model, called the randomly modulated periodicity (RMP) that allows one to capture the intrinsic variability of a cycle. As with a deterministic periodic signal the RMP can have a number of harmonics. The likelihood ratio test for this model when the amplitudes and phases are known is given in [M.J. Hinich, Signal Processing 83 (2003) 1349–1352]. A method for detecting a RMP whose amplitudes and phases are unknown random process plus a stationary noise process is addressed in this paper. The only assumption on the additive noise is that it has finite dependence and finite moments. Using simulations based on a simple RMP model we show a case where the new method can detect the signal when the signal is not detectable in a standard waterfall spectrogram display.

Keywords: Randomly modulated signal; Signal coherence spectrum; Periodic; Periodogram; Spectrogram; Waterfall display

1. Introduction

Consider the classic problem of detecting a sinusoid in additive noise. Suppose that the discrete-time sampled signal is of the form $y(t_n) = a\cos(2\pi f_o t_n + \theta) + e(t_n)$ where $e(t_n)$ denotes the noise at time $t_n = n\delta$ where δ is the sampling interval. Assume that the amplitude a and phase θ of the sinusoid are unknown parameters. If the frequency f_o is known and the null hypothesis is that $a = 0$ then the standard test of this hypothesis is to use the Fisher periodogram test [1]. The periodogram of a block of

*Corresponding author. Tel.: +1 512 232 7270; fax: +1 512 835 3259.
E-mail addresses: hinich@mail.la.utexas.edu (M.J. Hinich), p.wild@economics.uq.edu.au (P. Wild).

0165-1684/$ - see front matter
doi:10.1016/j.sigpro.2005.02.008

the signal is

$$I(k) = \frac{1}{N}\left|\sum_{n=0}^{N-1} y(t_n)\exp(-2\pi f_k t_n)\right|^2$$

where $f_k = k/T$ is the kth Fourier frequency and $T = N\delta$. In contrast for most signal processing applications, such as sonar and radio astronomy, a large peak at frequency f_o in the periodogram is assumed to be generated by a sinusoid in the signal at that frequency if there is a credible reason for the existence of a sinusoid at that frequency. In other words the usual signal processing practice of detecting sinusoids is not treated as a formal statistical problem.

All signals that appear to be periodic have some sort of variability from period to period regardless of how stable they appear to be in a data plot. A true sinusoidal time series is a deterministic function of time that never changes and thus has zero bandwidth around the sinusoid's frequency. A zero bandwidth is impossible in nature since all signals have some intrinsic variability over time.

In active sonar the outgoing acoustic pings are virtually the same from ping to ping. But each received sonar ping has some random modulation. The amount of ping to ping variation in the receive signal is surprisingly large. A passive sonar signal has a lot of modulation due to the scattering and reflections in the water.

Deterministic sinusoids are used to model cycles as a mathematical convenience. It is time to break away from this simplification in order to model the various periodic signals that are observed in fields ranging from biology, communications, acoustics, astronomy, and the various sciences.

Hinich [2] introduced a parametric statistical model, called the Randomly Modulated Periodicity (RMP) that allows one to capture the intrinsic variability of a cycle. As with a deterministic periodic signal the RMP can have a number of harmonics. The likelihood ratio test for this model when the amplitudes and phases are known is given in [3]. In that paper, the detection problem was structured around a simple null (gaussian white noise) and a simple alternative (a known sinusoid plus gaussian noise). The main result was that the optimal detector for this problem was a linear combination of the periodogram and a matched filter. The Fisher periodogram test was demonstrated to be sub-optimal if there was prior knowledge of the modulation.

In this paper we significantly extend this work by addressing the more realistic (and complicated) detection problem for a RMP whose amplitudes and phases are, themselves, complicated random processes. In contrast to the detection problem presented in [3], the alternative process in this paper is much more general. In fact this generalization is achieved by not actually requiring any specification of the nature of the modulations apart from requiring a joint density and finite dependence.

2. A randomly modulated periodicity

A discrete-time random process $x(t_n)$ is an RMP with period $T = N\delta$ and K harmonic frequencies $f_k = k/T$ if it is of the form

$$x(t_n) = s_0 + \frac{2}{N}\sum_{k=1}^{K}[(s_{1k} + u_{1k}(t_n))\cos(2\pi f_k t_n) + (s_{2k} + u_{2k}(t))\sin(2\pi f_k t_n)], \tag{2.1}$$

where s_{1k} and s_{2k} are constants. The modulation processes $\{u_{11}(t_1),\ldots,u_{1,N/2}(t_n),\ u_{21}(t_n),\ldots,u_{2,N/2}(t_n)\}$ are unknown random processes with zero means, finite cumulants and a joint distribution that has the following finite dependence property: $\{u_{jr}(t_1),\ldots,u_{jr}(t_m)\}$ and $\{u_{ks}(t'_1),\ldots,u_{ks}(t'_n)\}$ are independent if $t_m + D < t'_1$ for some D and all $j,k = 1,2$ and $r,s = 1,\ldots,N/2$ and all sample times. The modulations increase the bandwidth of the signal above the highest harmonic f_K. Therefore, the sampling frequency $1/\delta$ must be greater than twice the highest frequency of the signal in order to avoid aliasing.

Finite dependence is a strong mixing condition [4]. If $D \ll N$ then the modulations are approximately stationary within each period. Finite cumulants, finite dependence and $D \ll N$ are the only assumptions made about the modulations.

The process can be written as $x(t_n) = s(t_n) + u(t_n)$ where

$$s(t_n) = E[x(t_n)] = s_0 + \frac{2}{N}\sum_{k=1}^{N/2} [s_{1k}\cos(2\pi f_k t_n) + s_{2k}\sin(2\pi f_k t_n)] \quad (2.2)$$

and

$$u(t_n) = \frac{2}{N}\sum_{k=1}^{N/2} [u_{1k}\cos(2\pi f_k t_n) + u_{2k}\sin(2\pi f_k t_n)]. \quad (2.3)$$

Thus $s(t_n)$, the expected value of the signal $x(t_n)$, is a periodic function. The fixed coefficients s_{1k} and s_{2k} determines the shape of $s(t_n)$. If $s_{11} \neq 0$ or $s_{21} \neq 0$ then $s(t_n)$ is periodic with period T. If $s_{11} = 0$ and $s_{21} = 0$ but $s_{12} \neq 0$ or $s_{22} \neq 0$ then $s(t_n)$ is periodic with period $T/2$. If the first $k_o - 1$ s_{1k} and s_{2k} are zero but not the next then $s(t_n)$ is periodic with period T/k_o.

The RMP model is superficially similar to an AM or FM signal but the modulation amplitude can be much larger than the amplitude of the "carrier" (the mean periodicity). The bandwidth of an RMP can be large. It is a type of spread spectrum signal but it is generated by the mechanism underlying the periodic process and is not a communication signal for the applications we have in mind such as active and passive sonar signal processing.

3. Signal coherence spectrum

To provide a measure of the modulation relative to the underlying periodicity, Hinich [2] introduced a concept called the *signal coherence spectrum* (SIGCOH). This SIGCOH concept is extended in this paper to problem of detecting an RMP in additive stationary noise.

Suppose that the observed signal is $y(t_n) = s(t_n) + u(t_n) + e(t_n)$ where $s(t_n)$ and $u(t_n)$ are defined by expressions (2.2) and (2.3). Assume that the additive noise $e(t_n)$ is strictly stationary with finite dependence of span D and finite moments. Thus the combined noise and modulation signal $\kappa(t_n) = u(t_n) + e(t_n)$ satisfies finite dependence and is stationary within the observation range.

For each Fourier frequency $f_k = k/T$ the value of SIGCOH is

$$\gamma_y(k) = \sqrt{\frac{|s_k|^2}{|s_k|^2 + \sigma_\kappa^2(k)}}, \quad (3.1)$$

where $s_k = s_{1,k} + \mathrm{i}s_{2,k}$ is the amplitude of the kth sinusoid, $\sigma_\kappa^2(k) = E|K(k)|^2$ and

$$K(k) = \sum_{n=0}^{N-1} (u(t_n) + e(t_n))\exp(-\mathrm{i}2\pi f_k t_n) \quad (3.2)$$

is the discrete Fourier transform (DFT) of $\kappa(t_n) = u(t_n) + e(t_n)$.

The amplitude-to-noise standard deviation is

$$\rho_y(k) = \frac{|s_k|}{\sigma_\kappa(k)}$$

for frequency f_k. Thus it follows that

$$\rho_x^2(k) = \frac{\gamma_x^2(k)}{1 - \gamma_x^2(k)}.$$

Suppose that we know the fundamental period and we observe the signal over M such periods. The mth period is $\{y((m-1)T + t_n), n = 0, \ldots, N-1\}$. The estimator of $\rho_x^2(k)$ introduced by Hinich [2] is

$$\hat{\rho}_x^2(k) = \frac{|\bar{X}(k)|^2}{\sigma_\kappa^2(k)}, \quad (3.3)$$

where $\bar{Y}(k) = M^{-1}\sum_{m=1}^{M} Y_m(k)$ is the sample mean of $Y_m(m) = \sum_{n=0}^{N-1} y((m-1)T + t_n)\exp(-\mathrm{i}2\pi f_m t_n)$ and $\hat{\sigma}_\kappa^2(k) = M^{-1}\sum_{m=1}^{M} |Y_m(k) - \bar{Y}(k)|^2$ is the sample variance of the residual DFT $Y_m(k) - \bar{Y}(k)$. This estimator is consistent as $M \to \infty$.

If $D \ll N$ where $N = T/\delta$ then the distribution of $(M/N)\rho_x^2(k)$ is asymptotically χ^2 with two degrees-of-freedom with a noncentrality parameter $\lambda_k = (M/N)\rho_x^2(k)$ as $M \to \infty$ [5].

These $\chi_2^2(\lambda_k)$ statistics are asymptotically independently distributed over the frequency band. Thus the distribution of the sum statistic

$$S = \sum_{k=1}^{K} \frac{M}{N}\hat{\rho}_x^2(k) \quad (3.4)$$

is approximately chi-squared $\chi_K^2(\lambda)$ where $\lambda = \sum_{k=1}^{K} \lambda_k$ for large values of M.

These chi-squared $\chi_2^2(\lambda_k)$ statistics and S are used to detect the presence of a hidden periodicity in the signal as shown in the next section.

4. Signal detection

Suppose that we observe a signal $y(t_n)$. The null hypothesis is $y(t_n) = e(t_n)$ where the noise process $\{e(t_n)\}$ is strictly stationary. The alternative hypothesis is $y(t_n) = s(t_n) + u(t_n) + e(t_n)$. This alternative is a complicated nonparametric stochastic model. The standard optimal detection theory does not apply to this alternative to the stationary noise hypothesis.

Thus from the asymptotic results stated in the previous section the distribution of $(M/N)\hat{\rho}_x^2(k)$ is approximately chi-squared $\chi_2^2(\lambda_k)$ and S is approximately chi-squared $\chi_K^2(\lambda)$ for $\lambda = \sum_{k=1}^{K} \lambda_k$ when the signal is present and is central χ_2^2 and χ_K^2, respectively for the null hypothesis of noise alone.

The cumulative distribution function of a central χ^2 random variable is $F_\chi(x) = \Pr(\chi_2^2 < x)$. Thus it follows from basic probability theory that under the null hypothesis $U(k) = F_\chi(MN^{-1}\hat{\rho}_x^2(k))$ has an approximately uniform (0, 1) distribution for each k. We call the display of the $U(k)$ statistics a *signal coherence probability spectrum*. These probability values are also asymptotically independent as $M \to \infty$.

This method has been implemented by Hinich in a Fortran 95 program that is available upon request. This simple test augments the standard spectral method. In particular the S test statistic is simple to compute and easy to automate. Furthermore, the signal model is realistic when compared to the zero bandwidth harmonics underpinning standard theoretical periodic models. Moreover, the assumptions for the modulations are modest. Only stationarity and finite dependence is assumed for the additive noise process. There is no need to assume that the noise is gaussian.

The assumption that the fundamental frequency is known can be relaxed if there is a reasonable belief that there is a RMP in the data within a given band. In activating this procedure, the investigator is essentially making a sweep of trial fundamental frequencies over the band to find the maximum value of S and its frequency $f_{\max}$, and then computing the p-value tail probability of the maximum. While this sweep method formally violates the purity of hypothesis testing from a practical perspective, if the p-value of the maximum S is small, say 1.e-5, then any reasonable person would assume that an RMP has been detected with a fundamental frequency of $f_{\max}$.

The next section presents simulation results that show that this new method can detect weak RMP signals that are missed by the standard waterfall (spectrogram) approach used in sonar and certain geophysical signal processing applications. The simulations have to use precisely defined modulation processes that have a reasonable number of parameters. Making the simulations very complicated renders the simulations useless for making comparisons between alternative detection methods.

5. Simulation analysis

The model used in the simulations is

$$x(t_n) = \frac{2}{N}\sum_{k=1}^{K}[(1 + \sigma_u u_{1k}(t_n))\cos(2\pi f_k t_n) + (1 + \sigma_u u_{2k}(t_n))\sin(2\pi f_k t_n)] + e(t_n) \quad (5.1)$$

where the $\{u_{11}(t_1), \ldots, u_{1,N/2}(t_n), u_{21}(t_n), \ldots, u_{2,N/2}(t_n)\}$ are independently distributed random variables with zero means and the additive noise is pure white noise (i.i.d) with variance σ_e^2. Thus the parameters of this model are K, M, N, σ_e and σ_u.

In sonar signal processing the hydrophone signals are formed into beams by delay and sum beamforming. The beam signals are blocked into adjacent equal length sections of sampled data. A periodogram is computed for each block and the periodogram values are thresholded to eliminate as many spurious peaks as possible and yet reveal most peaks due to a sinusoid component. These periodograms are stacked into a data structure where the horizontal axis is frequency and the vertical axis is the start time of the block. The display of this data is called a waterfall display or a spectrogram. A periodic signal will show up as a set of harmonically related lines down the waterfall.

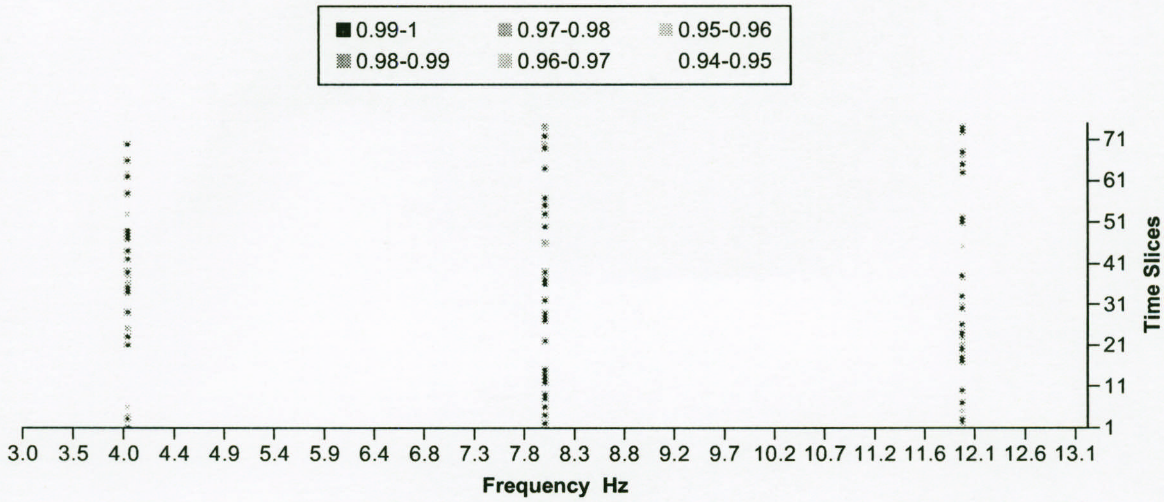

Fig. 1. RMP waterfall plot of normalized spectrograms—$\sigma_u = 20$ and $\sigma_e = 0$ and $0.95 < \text{peak values} < 1.0$.

Fig. 1 presents a waterfall spectrogram plot of 75 consecutive blocks of $M = 400$ frames of length $N = 50$ of artificial data generated by the model in (5.1) with $K = 2$ harmonics, $\sigma_u = 20$ and $\sigma_e = 0$ (no additive noise). The units for the signal were chosen so that the fundamental frequency was $f_1 = 4\,\text{Hz}$ and thus the two harmonics are $f_2 = 8$ and 12 Hz. The periodograms were divided by the maximum value for each block and the plot has a floor off 0.95 and so only values in the interval (0.95, 1.) are shown. The modulation fuzzes the waterfall lines but one can see that there is a periodicity in the signal.

A waterfall plot of the signal coherence probabilities for the same model is shown in Fig. 2. This plot also has a floor of 0.95 for the probabilities. The visual signal detection and harmonic analysis is sharper than for the spectrogram plot.

Fig. 3 is a spectrogram waterfall plot of the signal with an additive noise standard deviation of $\sigma_e = 20$ which yields a signal-to-noise ratio (SNR) of $-44\,\text{dB}$ using the variances of the two frequency bins around each of the three harmonic frequencies. It is apparent from inspection of this figure that the signal is not detectable in this plot.

Fig. 4 displays the waterfall plot of the signal coherence probabilities for the $-44\,\text{dB}$ signal. It is now evident that the signal is detectable in this plot.

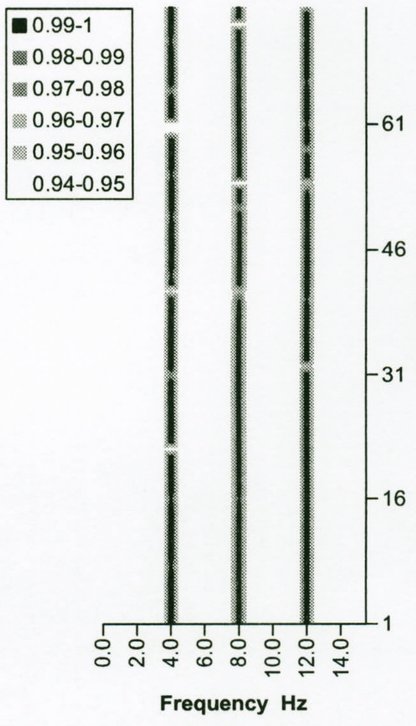

Fig. 2. Waterfall plot of signal coherence probabilities—$\sigma_u = 20$ and $\sigma_e = 0$ and $0.95 < \text{peak values} < 1.0$.

In order to estimate the false alarm probability and detection probability of the χ^2 S-based test we ran 4000 replications using both a 1% and 5%

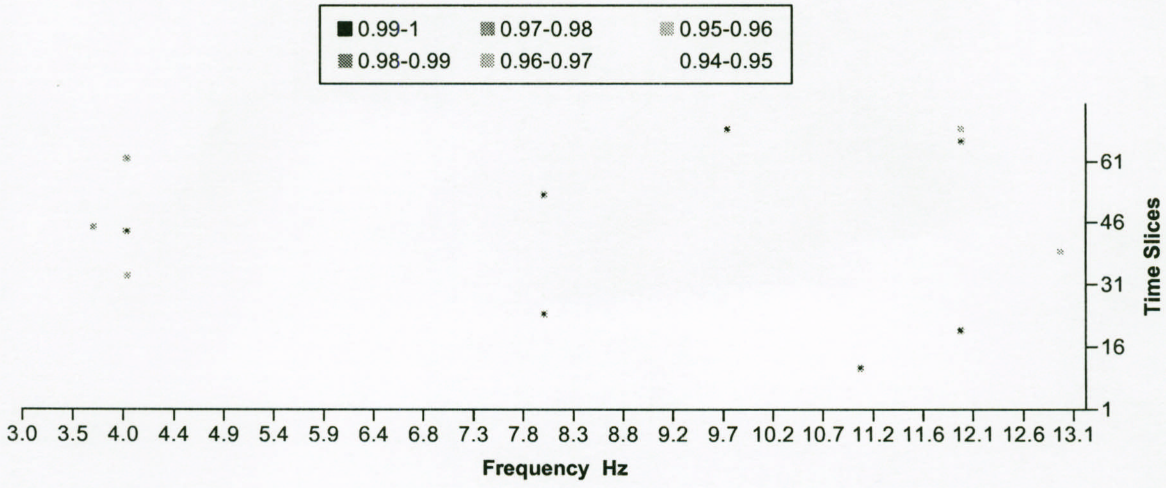

Fig. 3. RMP waterfall plot of normalized spectrograms—$f_1 = 4\,\text{Hz}$, $\sigma_u = 20$, $\sigma_e = 20$, SNR $= -44\,\text{dB}$ and $0.95 < \text{peak values} < 1.0$.

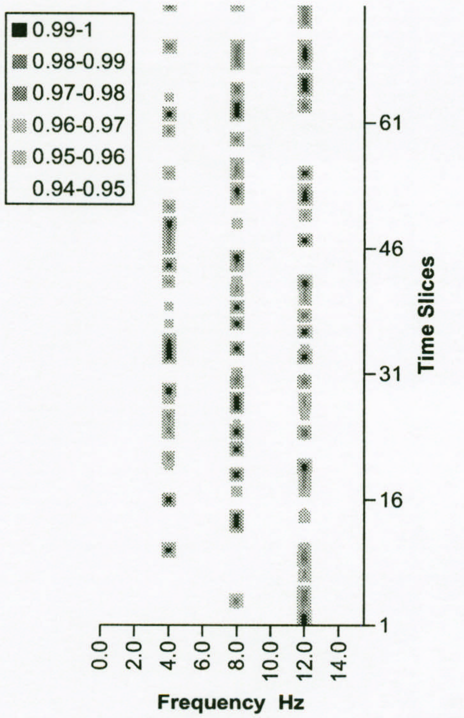

Fig. 4. Waterfall plot of signal coherence probabilities—$f_1 = 4\,\text{Hz}$, $\sigma_u = 20$, $\sigma_e = 20$, SNR $= -44\,\text{dB}$ and $0.95 < \text{peak values} < 1.0$.

threshold. For the null hypothesis of only noise the estimated false alarm probabilities were 0.013 and 0.057, respectively. The standard errors of the 1% estimate and 5% estimate are 0.0016 and 0.0034, respectively. Thus the estimates are not statistically different from the target false alarm probabilities.

The estimated probability of detecting the $-44\,\text{dB}$ signal ($\sigma_e = 20$) at the 1% level is 0.483 for 4000 replications. The estimated probability of detection at the 5% level is 0.709. These results exhibit the power of the RMP test for detecting realistic periodic signals in noise.

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