Randomly Modulated Periodic Signals

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Rotating Cylinder Data

Fluid Nonlinearity

Amplitude
Hourly Alberta Electricity Demand

Canadian $/US $ Rates of Return

Bassoon Note
Flute Note
Sound from a Power Boat

Amplitude

Time
Definition of a RMP

A signal is called a \textit{randomly modulated periodicity} with period $T=\nu\delta$ if it is of the form

$$x(t_n) = \mu_0 + N^{-1} \sum_{k=1}^{K} \left[ \left( s_{1k} + u_{1k}(t_n) \right) \cos(2\pi f_k t_n) + \left( s_{2k} + u_{2k}(t_n) \right) \sin(2\pi f_k t_n) \right]$$

$$f_k = \frac{k}{T} \quad t_n = n\delta \quad Eu_{1k}(t_n) = Eu_{2k}(t_n) = 0$$

for each $k = 1, \cdots, K$ where $K \leq \frac{T}{2\delta}$
**Random Modulations**

The vector of the $K$ modulations

$$\mathbf{u}(t_n) = \{u_{1k}(t_n), u_{2k}(t_n) : k = 1, \ldots, K\}$$

are jointly dependent random processes

for all $0 < t_1 < \cdots < t_m < T$
**Finite Dependence**

Condition needed to ensure that averaging over frames yields asymptotically gaussian estimates

\[
\{ u(t_1), \ldots, u(t_m) \} \quad \& \quad \{ u(t'_1), \ldots, u(t'_r) \}
\]

are independently distributed if

\[
t_m + D < t'_1 \quad \text{for some } D \quad \& \quad \text{and all }
\]

\[
t_1 < \cdots < t_m \quad \& \quad t'_1 < \cdots < t'_n
\]
Fourier Series for Components

Thus \( x(t_n) = s(t_n) + u(t_n) \) where

\[
s(t_n) = s_0 + N^{-1} \sum_{k=1}^{K} \left[ s_{1k} \cos(2\pi f_k t_n) + s_{2k} \sin(2\pi f_k t_n) \right]
\]

\[
u(t_n) = N^{-1} \sum_{k=1}^{K} \left[ u_{1k} \cos(2\pi f_k t_n) + u_{2k} \sin(2\pi f_k t_n) \right]
\]
Signal Plus Noise

\[ s(t_n) \text{ is the mean of } x(t) \]

\[ \{u(t_n)\} \text{ has a periodic joint distribution} \]

The modulation is part of the signal

*It is not measurement noise*
Artificial Data Examples

\[ x(t_n) = s_0 + N^{-1} \sum_{k=1}^{K} \left[ (1 + \sigma u_{1k}(t_n)) \cos(2\pi f_k t_n) + (1 + \sigma u_{2k}(t_n)) \sin(2\pi f_k t_n) \right] \]

\[ u_{1k}(t_n) = \rho u_{1k}(t_n - T) + e_1(t_n) \]

\[ u_{2k}(t_n) = \rho u_{2k}(t_n - T) + e_2(t_n) \]
Five Standard Deviations

10 Harmonics  Modulation $\sigma = 5$  $\rho = 0.9$  Frame=100
Three Standard Deviations

10 Harmonics  Modulation $\sigma = 3$  $\rho = 0.9$  Frame=100
Two Standard Deviations

10 Harmonics  Modulation $\sigma = 2$  $\rho = 0.9$  Frame =100
One Standard Deviation

Randomly Modulated Pulses
10 Harmonics  Modulation $\sigma = 1$  $\rho = 0.9$  Frame = 100
No Correlation in the Modulation

Four Randomly Modulated Pulses  Frame = 100  \( \sigma=5 \)
The data block is divided into $M$ frames of length $T$

$T$ is chosen by the user to be the period of the periodic component

The $t$-th observation in the $m$th frame is

$$x((m-1)T + n\delta) \quad n = 0, \ldots, N-1$$
Frame Rate Synchronization

The frame length $T$ is chosen by the user to be the hypothetical period of the randomly modulated periodic signal.

If $T$ is not an integer multiple of the true period then coherence is lost.
\[ \gamma_x(k) = \sqrt{\frac{|S_k|^2}{|S_k|^2 + \sigma_u^2(k)}} \]

The signal-to-noise ratio is

\[ \rho_x(k) = |S_k|^2 \sigma_u^{-2}(k) \]

\[ \rho_x(k) = \frac{\gamma_x^2(k)}{1 - \gamma_x^2(k)} \]
Estimating Signal Coherence

\{ \hat{x}(t_n) : n = 0, \ldots, N - 1 \} \text{ is the mean frame averaged over the } M \text{ frames}

\hat{X}(k) = \sum_{n=0}^{N-1} \hat{x}(t_n) \exp(-i 2 \pi f_k t_n)

\hat{y}_x(k) = \sqrt{\frac{|\hat{X}(k)|^2}{|\hat{X}(k)|^2 + \hat{\sigma}^2_u(k)}}

\hat{\sigma}^2_u(k) = M^{-1} \sum_{m=1}^{M} |X_m(k) - \hat{X}(k)|^2
Statistical Measure of Modulation SNR

\[ Z(k) = \frac{M}{N} \frac{\left| \hat{X}(k) \right|^2}{\hat{\sigma}_x^2(k)} \]

\[ = \frac{M}{N} \hat{\rho}_x^2(k) \]

\[ \hat{\sigma}_x^2(k) = \left| \hat{X}(k) \right|^2 + \hat{\sigma}_u^2(k) \]
Chi Squared Statistic

\[ Z(k) = \frac{M}{N} \hat{\rho}_x^2(k) \]

If the modulation is stationary the distribution of each \( Z(k) \) is approximately \( \chi^2_2(\lambda_k) \) & they are independently distributed.

\[ \lambda_k = \frac{M}{N} \rho_x^2(k) \]
Spectrum of the Variance

$s(t)$ - a stationary random process

Fourier Series Expansion

$$s(t_n) = a_0 + 2 \sum_{k=1}^{K} a_k \cos(2\pi f_k t_n) + 2 \sum_{k=1}^{K} b_k \sin(2\pi f_k t_n)$$

$$Var(a_k) = Var(b_k) \propto S(f_k) - Spectrum$$

$$\sigma_s^2 = \sum_{k=1}^{K} S(f_k)$$
Power & Signal Coherence Spectra - Demand

Power & Signal Coherence Spectra of the Residuals from an AR(12) Fit of the Alberta Electricity Hourly Spot Demand

dB

Spectrum  Coherence

Signal Coherence

Period in Hours

168.0 24.0 12.9 8.8 6.7 5.4 4.5 3.9 3.4 3.1 2.8 2.5 2.3 2.1

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
Canadian$/US$ Daily Data Spectra

Coherence Spectrum

Spectrum - US$/Canadian$  Coherence Spectrum

dB

Coherence Probability

Period in Days
Bicorrelations of a Random Signal

\[ c_{xxx}(t_1, t_2, t_3) = E \left[ x(t_1) x(t_2) x(t_3) \right] \]

If \( \{x(t)\} \) is stationary then

\[ c_{xxx}(t_1, t_2, t_3) = E \left[ x(t + t_1 - t_3) x(t + t_2 - t_3) x(t) \right] \]

\[ c_{xxx}(\tau_1, \tau_2) = E \left[ x(t) x(t + \tau_1) x(t + \tau_2) \right] \]
The Bispectrum

\[ S_{xxx}(f_1, f_2) = \int_{-\infty}^{\infty} c_{xxx}(\tau_1, \tau_2) \exp[-i2\pi(f_1\tau_1 + f_2\tau_2)] d\tau_1 d\tau_2 \]

If the noise is \textit{gaussian} then

\[ S_{xxx}(f_1, f_2) = 0 \]
Example of a Simple Nonlinear Model

\[
x(t_n) + a_1(x(t_{n-2}))x(t_{n-1}) + a_2(x(t_{n-2}))x(t_{n-2}) = \sigma u(t_n)
\]

\[
a_2(x(t_{n-2})) = e^{-c(x(t_{n-2}))},
\]

\[
a_1(x(t_{n-2})) = -2a_2(x(t_{n-2}))\cos\omega(x(t_{n-2}))
\]

\[
c(x(t_{n-2})) = c(1 + \delta x^2(t_{n-2}))
\]

\[
\omega(x(t_{n-2})) = \omega(1 + \delta x^2(t_{n-2}))
\]
Nonlinear Model - Uniform Input

\[ \sigma = 0.5, \quad c = 0.05, \quad \omega = 0.2\pi, \quad \delta = 0.5 \]
Phase Plot of the Nonlinear Impulse Response

Nonlinear Impulse Response Phase Plot

c = 0.05  \delta = 0.5
Phase Plot of the Linear Impulse Response

Phase Plot of Linear Impulse Response
\( c = 0.05 \), \( \omega = 0.2\pi \), \( \delta = 0 \)
Correlations Functions of \( x(t) \) & \(|x(t)|\)

Exponential Input \( \sigma = 0.5 \), \( c = 0.05 \), \( \omega = 0.2\pi \), \( \delta = 0.5 \)
Bispectrum - Nonlinear AR(2) Signal

Bispectrum of Nonlinear AR(2) \(c=0.05\) \(\delta=0.02\) \(f=0.2\)