On the Principal Domain of the Discrete Bispectrum of a Stationary Signal

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***Abstract*— This paper presents a simplifying, yet general approach to determining the symmetry structure of a bispectrum. The principal domain (PD) of the bispectrum and its region of positive support are derived in a way that illuminates the controversy surrounding the triangle in the PD, which is called the outer triangle (OT) by Hinich and Wolinsky, where the bispectrum is zero for a stationary random sampled process that is not aliased. The basic statistical issues of testing for nonzero bispectral structure are reviewed.**

I. Introduction

HIGHER order spectral analysis is considerably more complicated than spectral analysis. Going from one to two or more frequencies introduces many complicating technical issues. The symmetries of the bispectrum are not obvious, and those of the trispectrum are even more complicated. This paper introduces a fresh approach to bispectral analysis.

The statistical properties of higher order spectral estimates are more complex than spectral estimates. For example, the variances of the bispectrum depend on the trispectrum and the sixth-order cumulant spectrum, whereas the variances of spectral estimates depend on the power spectrum and the trispectrum. These variance parameters play a crucial role in computing the asymptotic properties (both bispectrum and trispectrum) based on the tests for nonlinear and nonGaussian structure that have received considerable interest and application in a number of fields, including engineering signal processing [10], [11].

This paper presents a window to the technical issues inherent in the successful application of higher order spectra for signal analysis.

One of the major benefits in using a higher than second-order cumulant is the possibility to discriminate between Gaussian and nonGaussian random signals. Since the bispectrum of a Gaussian sequence is zero over the entire principal domain, a test for Gaussianity based on the estimated bispectrum has been suggested [4], [14]. This test is well accepted by the signal processing community and has been used for signal detection problems (e.g., [6], [9]).

Hinich and Wolinsky [5] have studied the discrete bispectrum of a sequence created by sampling a stationary, random process. They have shown that the principal domain can be divided into two triangles: the inner triangle (IT) and the outer triangle (OT). If the sampling rate agrees with the Nyquist rule, the discrete bispectrum over the OT is identically zero. Based on this observation, they suggested a test for aliasing [5]. Sharfer and Messer [13] used it for testing for jitter in the sampling clock. This test, as well as a bispectral-based test for a transient coherent signal in stationary noise [7] and other advanced tests based on zero bispectrum over subregions of the principal domain, has been criticized lately on several occasions (e.g., [3], [12], [15]). In this paper, we prove, in a way different from [5], that the bispectrum of a discrete sequence is zero over the OT under certain conditions. We study these conditions, and we explain how each of them can be related to properties of the tested signal. Thus, its violation can be tested using estimates of the discrete bispectrum over the OT.

II. Theoretical Background

Assume first that $x(t)$ is a real, zero-mean[1], continuous **A1) random process**. Define $c(t_1, t_2, t_3) = E\{x(t_1)x(t_2)x(t_3)\}$, and assume it is finite. The 3-D Fourier transform of $c(t_1, t_2, t_3)$ (which is the third-order simple cumulant of $x(t)$) is the third-order cumulant spectrum of $x(t)$

$$\mathrm{CS}(\omega_1, \omega_2, \omega_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(t_1, t_2, t_3) \times e^{-j\omega_1 t_1} e^{-j\omega_2 t_2} e^{-j\omega_3 t_3}\, dt_1\, dt_2\, dt_3. \tag{1}$$

If $x(t)$ is at least third-order **A2) stationary**, its third-order cumulant spectrum is only a function of two variables $c(t_1, t_2, t_3) = c(t_1 - t_2,\ t_2 - t_3) = c(\tau_1, \tau_2)$. A 2-D Fourier transform of this bicovariance function $c(\tau_1, \tau_2) = E\{x(t)x(t-\tau_1)x(t-\tau_2)\}$, can then be applied. The resultant 2-D function of ω_1 and ω_2 is the **bispectrum** of the stationary, random signal $x(t)$

$$B(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(\tau_1, \tau_2) e^{-j\omega_1\tau_1} e^{-j\omega_2\tau_2}\, d\tau_1 d\tau_2. \tag{2}$$

For a stationary $x(t)$, the third-order cumulant spectrum must satisfy

$$\mathrm{CS}(\omega_1, \omega_2, \omega_3) = \delta(\omega_1 + \omega_2 + \omega_3) B(\omega_1, \omega_2) \tag{3}$$

[1] The assumptions of zero mean and real signal can be relaxed. We use them only for simplicity.

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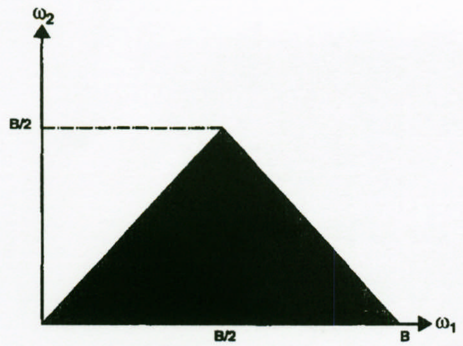

Fig. 1. PD of the continuous bispectrum of a bandlimited signal.

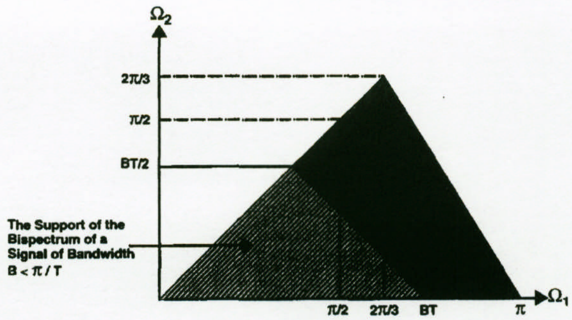

Fig. 2. PD of the discrete bispectrum.

where $\delta(\omega)$ is the Dirac delta function. Notice that the third-order cumulant spectrum is well defined for any signal. If $x(t)$ happens to be a stationary, random signal, it becomes identically zero anywhere in $(\omega_1, \omega_2, \omega_3)$ but over the plane $(\omega_1 + \omega_2 + \omega_3)$. Using the Cramer representation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t}\, dX(\omega). \tag{4}$$

Then, the third-order cumulant spectrum of (1) is given by

$$\mathrm{CS}(\omega_1, \omega_2, \omega_3)\, d\omega_1 d\omega_2 d\omega_3 = E\{dX(\omega_1) dX(\omega_2) dX(\omega_3)\}. \tag{5}$$

Assume that $x(t)$ is **A3) bandlimited** to $B = 2\pi W$; then, $dX(\omega) = 0$ for $|\omega| > B$ and, from (5), $\mathrm{CS}(\omega_1, \omega_2, \omega_3) = 0$ outside the cube $\{|\omega_k| \leq B; k = 1, 2, 3\}$. We now derive the principal domain of the third-order cumulant spectrum for bandlimited signals. From (1), we see that $\mathrm{CS}(\omega_1, \omega_2, \omega_3)$ is invariant to permutation of the frequencies. Therefore, the cube can be divided into six equal volume regions defined by

$$\begin{aligned} &\omega_1 < \omega_2 < \omega_3 \\ &\omega_1 < \omega_3 < \omega_2 \\ &\omega_2 < \omega_1 < \omega_3 \\ &\omega_2 < \omega_3 < \omega_1 \\ &\omega_3 < \omega_1 < \omega_2 \\ &\omega_3 < \omega_2 < \omega_1. \end{aligned} \tag{6}$$

We further assume that $x(t)$ is a **real signal**. As such, $\mathrm{CS}(\omega_1, \omega_2, \omega_3) = \mathrm{CS}^*(-\omega_1, -\omega_2, -\omega_3)$. For each triplet $(\omega_1, \omega_2, \omega_3)$ in the cube $|\omega_1| < B, |\omega_2| < B, |\omega_3| < B$, there exist 11 images determined by the composition of the permutation and the sign change resulting from the conjugate operation. To establish the boundaries of a nonredundant region, we take the intersection of a double pyramid of (5) and the orthants $(+++)$, $(++-)$, $(+-+), \ldots, (---)$. The symbol $(+++)$ stands for all positive frequencies, the symbol $(++-)$ stands for ω_1 and ω_2 positive, whereas ω_3 is negative, etc. Because of the sign invariant, each of the plus signs can be replaced by a minus sign, and vice versa. Composing it with the permutation of the frequencies, only two orthants, say $(+++)$ and $(++-)$, are sufficient for all eight of them. Using one of the pyramids of (5), say, the last one, its intersection with $(+++)$ is simply $\omega_3 < \omega_2 < \omega_1$. Its intersection with $(++-)$ is $\omega_3 < 0 < \omega_2 < \omega_1$. A principal domain of the third-order cumulant spectrum of a bandlimited signal is then given by

$$\{\omega_3 < \omega_2 < \omega_1\} \cup \{-B < \omega_3 < B\}. \tag{7}$$

From (3), we have that for a stationary signal, the third-order cumulant spectrum is zero in most of the volume of a principal domain (see Fig. 2). The frequencies of nonzero values are on the intersection of a volume that is a principal domain with the plane $\omega_1 + \omega_2 + \omega_3 = 0$. The result is the support of the bispectrum. By substituting $-\omega_3 = \omega_2 + \omega_1$ in (7), we see that this plane does not intersect the pyramid $0 < \omega_3 < \omega_2 < \omega_1$. Its intersection with $\omega_3 < 0 < \omega_2 < \omega_1 < B$ results in the triangle (see Fig. 1)

$$(\omega_1, \omega_2) \in \{0 < \omega_2 < \omega_1 < B\} \cup \{\omega_1 + \omega_2 < B\}. \tag{8}$$

This is actually the PD of the continuous bispectrum for a bandlimited signal.

Each of the 12 possible principal domains of the third-order cumulant spectrum in the cube $\{|\omega_k| \leq B;\ k = 1, 2, 3\}$ intersects the plane $\omega_3 + \omega_2 + \omega_1 = 0$ in a different manner. While some intersections result in a triangle (as in our example), others result in two disjoint triangles in the plane (ω_1, ω_2).

The next step is to assume that the bandlimited, continuous signal is sampled by a sampling rate that satisfies the Nyquist condition $2\pi f_s \geq 2B (f_s \geq 2W)$. The discrete third-order cumulant spectrum of the sampled signal $x(n) = x(n/f_s)$ then satisfies

$$\begin{aligned} &\mathrm{CS}_d(\Omega_1, \Omega_2, \Omega_3) \\ &= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} c(n_1, n_2, n_3) e^{-j\Omega_1 n_1} \\ &\quad e^{-j\Omega_2 n_2} e^{-j\Omega_3 n_3} \\ &= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} \mathrm{CS}\left(\frac{\omega_1}{2\pi f_s} + n_1, \frac{\omega_2}{2\pi f_s} + n_2, \right. \\ &\qquad\qquad \left. \frac{\omega_3}{2\pi f_s} + n_3 \right) \end{aligned} \tag{9}$$

where $\Omega_i = \frac{\omega_i}{2\pi f_s}$. For a stationary signal, it follows from (3) that

$$\begin{aligned}
\mathrm{CS}_d(\Omega_1, \Omega_2, \Omega_3) &= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} \mathrm{CS}\bigg(\frac{\omega_1}{2\pi f_s} + n_1, \frac{\omega_2}{2\pi f_s} + n_2, \\
&\qquad \frac{\omega_3}{2\pi f_s} + n_3\bigg) \\
&= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} \delta(\Omega_1 + n_1 + \Omega_2 + n_2 + \Omega_3 + n_3) \\
&\qquad \times B(\Omega_1 + n_1, \Omega_2 + n_2). \qquad (10)
\end{aligned}$$

The discrete third-order cumulant spectrum consists of a mosaic of cubes of dimension 2π. Each one of the cubes in this infinite mosaic is made of the 12 volume regions that are a periodic extension of the principal domains of the continuous third-order cumulant spectrum.

For a stationary, bandlimited random signal where bandwidth $B \leq \pi f_s$, f_s is the sampling rate, the discrete cumulant spectrum is nonzero only over one plane in each of the cubes. For the nominal cube (about the origin), this plane is $\Omega_1 + \Omega_2 + \Omega_3 = 0$ (Fig. 2). For other cubes, the plane over which the cumulant spectrum is nonzero is $\Omega_1 + \Omega_2 + \Omega_3 = n$, where n is any nonzero integer. This follows from the last equation since the delta function is nonzero for any triplet $(\Omega_1,\ \Omega_2,\ \Omega_3)$ that sums up to an integer. Notice, however, that only the intersection of the plane $\Omega_1 + \Omega_2 + \Omega_3 = n$, $n \neq 0$ with the origin of a cube about $\Omega_1 = -n_1$, $\Omega_2 = -n_2$, $\Omega_3 = -n_3, n_1 + n_2 + n_3 = n$ is an image of the support of the bispectrum. On the intersection of such a plane with the other cubes, the cumulant spectrum is zero. For example, the nominal cube may be intersected by planes $\Omega_1 + \Omega_2 + \Omega_3 = n, n \neq 0$, which are parallel to the plane $\Omega_1 + \Omega_2 + \Omega_3 = 0$ on which, and only on which, the cumulant spectrum is nonzero[2].

A PD of the discrete bispectrum is given by [2] (see Fig. 3)

$$(\Omega_1, \Omega_2) \in \{2\Omega_1 + \Omega_2 < 2WT\} \cup \{0 < \Omega_2 < \Omega_1\}. \qquad (11)$$

By comparing it with the principal domain of the continuous, bandlimited bispectrum (8), we see that an extra trapezoid is added to the principal domain due to the sampling of the continuous, bandlimited process. The trapezoid becomes a triangle if $f_s = 2W$. We now show that this extra triangle, which is denoted as the OT [5], is the intersection of the plane $\Omega_1 + \Omega_2 + \Omega_3 = 1$ with the nominal cube of the discrete third-order cumulant spectrum. Thus, the discrete cumulant spectra is zero over this triangle, provided that the sampled process is stationary and that the sampling interval $T = 1/f_s$ satisfies $T \geq \frac{1}{2W}$ so that there is no aliasing.

Part of a PD of the discrete third-order cumulant spectrum is the composition of the volume region $-WT < \Omega_2 < \Omega_1 < \Omega_3 < WT$ with the orthant $(+\,+\,+)$, which leaves us with the pyramid $0 < \Omega_2 < \Omega_1 < \Omega_3 < WT$. Its intersection with the plane $\Omega_1 + \Omega_2 + \Omega_3 = 2WT$ creates a triangle in the (Ω_1, Ω_2) plane that is bounded by the axis and by the line $2\Omega_1 + \Omega_2 = 2WT$, which defines the OT in Fig. 3. We therefore reproved the result of [5], saying that *a principal domain of the discrete bispectrum of a time series that results from nonaliased sampling of a bandlimited, stationary random process is divided into two regions: the IT, which is the support of the continuous bispectrum* (8) *for* $\Omega_1 = \omega_1 T$ *and* $\Omega_2 = \omega_2 T$, *and the* OT *over which the discrete bispectrum is inherently zero.*

[2]If $f_s < 2W$, then the cubic mosaic consists of overlapping cubes of dimension $2B/f_s > 2\pi$. Therefore, each subcube of dimension 2π is intersected by planes of other cubes over which the cumulant spectrum is nonzero.

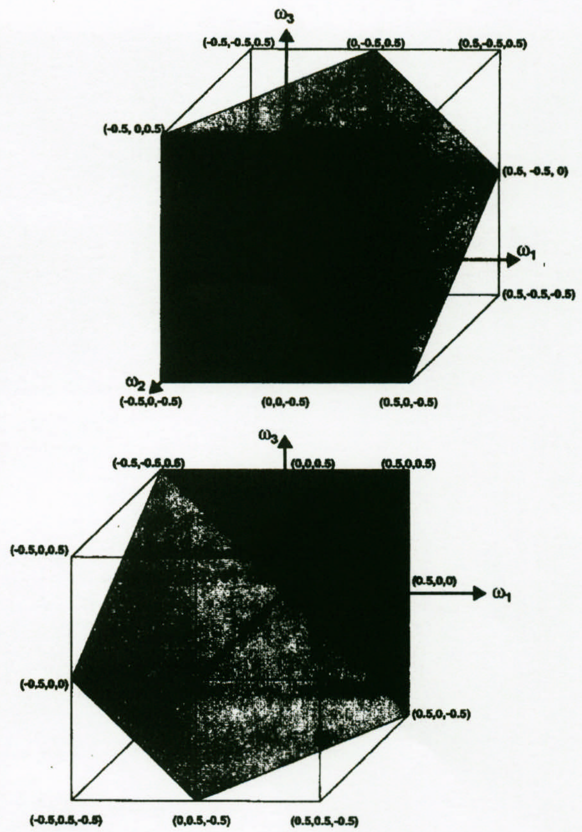

Fig. 3. Plane $\omega_1 + \omega_2 + \omega_3 = 0$.

The discrete bispectrum is related to the continuous bispectrum via

$$\begin{aligned}
B_d(\Omega_1, \Omega_2) &= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} c(m_1, m_2) e^{-j\Omega_1 m_1} e^{-j\Omega_2 m_2} \\
&= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} B\bigg(\frac{\omega_1}{2\pi f_s} + m_1, \frac{\omega_2}{2\pi f_s} + m_2\bigg). \qquad (12)
\end{aligned}$$

A PD of it is given by (11) and is depicted in Fig. 3.

Applying the discrete Fourier transform on the sampled signal $x(n) = x(t)|_{t=nt}$, one gets $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\Omega}$, which is a periodic function of Ω with a period on 2π. In addition, since $x(t)$ is bandlimited, $X(\Omega) = 0$ for $WT < |\Omega| < \pi$. Define $G(\Omega_1, \Omega_2) = X(\Omega_1)X(\Omega_2)X^*(\Omega_1 + \Omega_2)$; then, the PD of $G(\Omega_1, \Omega_2)$ is given by (11); therefore, it is the same as the PD of the discrete bispectrum. Notice, however, that while $G(\Omega_1, \Omega_2)$ is defined for **any** sequence $x(n)$, the bispectrum is only defined for stationary, random signals. We make no claim that $G(\Omega_1, \Omega_2)$ is zero over the OT, which is part of its PD. Our claim is valid only for the discrete bispectrum, which is not $G(\Omega_1, \Omega_2)$. Notice that the "counter examples" given in [3], [12], and [15] to show that the claim that the bispectrum over the OT is zero actually derive $G(\Omega_1, \Omega_2)$ for deterministic signals and show that over OT, it is nonzero and, therefore, are irrelevant to the claim.

In the next section, we show that estimation of the discrete bispectrum of a stationary, random signal is based on calculation of $G(\Omega_1, \Omega_2)$ at discrete frequencies for a finite sample of a continuous, bandlimited, stationary random signal.

Consider now a finite sample of the process $\{x(0), x(1), \ldots, x(N-1)\}$. The discrete Fourier transform of this sequence is $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$. If the signal $x(t)$ is stationary, then [1], [8] for (k_1, k_2, k_3) in the nominal cube

$$\begin{aligned} &E\{X(k_1)X(k_2)X(k_3)\} \\ &\quad = NB_d(k_1, k_2) + O(1) \qquad \forall\, k_1 + k_2 + k_3 = 0 \\ &\quad = O(1). \end{aligned} \tag{13}$$

Otherwise, $B_d(k_1, k_2)$ is the discrete bispectrum of (12) at $\Omega_i = \frac{2\pi}{N} k_i$.

III. Testing for Zero Bispectrum Over the OT

The discrete bispectrum over the OT has been proven to be zero if the signal satisfies the following three conditions:

1) It is a random signal.
2) It is a stationary signal.
3) It has been sampled without aliasing.

We present a statistical test to decide if the true bispectrum of a sampled signal is zero for all bifrequencies in the OT. If the test statistic is significant, we can infer that at least one of the assumptions does not hold.

Given a sample of size N from a continuous process, one can estimate the discrete bispectrum of the process. Based on (13), a nonparametric estimate is related to the biperiodigram of the data. If the spectrum of the continuous process $S(\omega)$ is known to be smooth over a frequency band not smaller than Δ_N, then the variance of the estimate of the discrete bispectrum of a stationary, random signal is

$$\mathrm{Var}\{\hat{B}_d(k_1, k_2)\} = \frac{S\big(2\pi f_s \frac{k_1}{N}\big) S\big(2\pi f_s \frac{k_2}{N}\big) S\big(2\pi f_s \frac{k_1+k_2}{N}\big)}{\Delta_N^2 N} \tag{14}$$

where $|k_1| \leq N$, $|k_2| \leq N$, and the sampled signal is assumed to satisfy a **A4) mixing condition** [1]. Finite memory of the random process is sufficient for it to satisfy the mixing conditions. Consistency of the estimate is, therefore, guaranteed if $\Delta_N^2 N$ goes to infinity while Δ_N goes to zero. For consistency, it is therefore sufficient to choose $\Delta_N = N^{-c}$, where $0 < c < 0.5$.

There are several ways to construct bispectrum consistent estimates from the biperiodigram by limiting the resolution bandwidth. We divide the data record into P frames, each of L samples. As such, $\Delta_N = \frac{1}{L}$ is the resolution bandwidth. For consistency, we take $L = N^c$, where $0 < c < 0.5$.

Define

$$G_p(k_1, k_2) = X_p(k_1) X_p(k_2) X_p^*(k_1 + k_2) \tag{15}$$

where $X_p(k) = \sum_{n=(p-1)L+1}^{pL} x(n) e^{-j2\pi kn/N}$, which is a consistent estimator of the discrete bispectrum of the random, stationary, bandlimited signal of finite memory that is properly sampled, is then given by

$$\hat{B}_d(k_1, k_2) = \frac{1}{P} \sum_{p=1}^{P} G_p(k_1, k_2) \tag{16}$$

where $2k_1 + k_2 < N$ and $0 < k_2 < k_1 < N$, and $P = \frac{N}{L} = N^{1-c}$. The bispectrum estimates are asymptotically complex normal and, at different bifrequencies, are asymptotically uncorrelated. Define

$$Y(k_1, k_2) = \frac{(\hat{B}_d(k_1, k_2) - B_d(k_1, k_2)) \Delta_N \sqrt{N}}{\sqrt{S(k_1) S(k_2) S(k_3)}}. \tag{17}$$

Then, asymptotically, $Y(k_1, k_2)$ for $(k_1, k_2) \in$ PD are i.i.d standard complex-normal random variables for a finite set of bifrequencies that are the limit points of $\{\frac{k_1(N)}{N}, \frac{k_2(N)}{N}\}$ (see [1] for details). This motivates the use of the statistic

$$z = \frac{\sum_{(k_1, k_2) \in \mathrm{OT}} \left(|Y(k_1, k_2)|^2 - 2\right)}{2\sqrt{\frac{L^2}{48}}} \tag{18}$$

to test for zero bispectrum over the OT, where Y is derived from (17) with $B_d(k_1, k_2) = 0$. $L^2/48$ is the number of the estimated bifrequencies in the OT. The hypothesis is rejected if z is larger than a prespecified threshold. For determination of the power of the test and to evaluate its performance, z is assumed to be a standard, normal random variable. This is true if $\{Y(k_1, k_2)\}_{(k_1, k_2) \in \mathrm{PD}}$ are i.i.d standard complex-normal random variables. The limiting distribution of the $Y(k_1, k_2)$ as well as the sum of squared Y_s involves a precise and complicated technical analysis. It can be shown that if $L = N^c$, where $0 < c < 0.5$, the test statistic is asymptotically $N(0, 1)$, but that in itself is insufficient to justify the use of the large sample approximation. The rate of convergence depends on the trispectrum of the process [8]. If the process is not too kurtotic, the test statistic is well approximated by the standard normal distribution $N(0, 1)$.

IV. Conclusion

In this paper, we present a method for evaluating the principal domain of the discrete bispectrum of a stationary, bandlimited random signal from its third-order cumulant spectrum. The method presented here can be generalized for the evaluation of polyspectrum of any order K using the cumulant spectrum of order $K+1$. The analysis shows that the principal domain of a discrete ployspectrum of order higher than one consists of a subregion over which the discrete ploysp ectrum of a stationary, bandlimited random signal is inherently zero, provided it was properly sampled. This region is referred to as the OT.

We then present a test of the (null) hypothesis that the discrete bispectrum is zero over the OT. We first split the data into P frames of L samples such that $L < \sqrt{N}$, where N is the total number of samples. Over each frame, we apply the fast Fourier transform (FFT) of the data, and we apply (15). The discrete bispectrum estimates are then given by (16) from which (17) is derived for $B_d(k_1, k_2) = 0, (k_1, k_2) \in$ OT. The test statistic z is then computed from (18) and is compared with a threshold. If z is larger than the threshold, the null hypothesis is rejected; therefore, we infer that the bispectrum over the OT is not zero. This indicates that at least one of the assumptions A1)–A4) is not satisfied, namely, we havwe the following:

1) The signal is not random.
2) The signal is random but nonstationary.
3) The signal is random and stationary but was not properly sampled (so the sampled signal is aliased).
4) The signal is random, stationary, bandlimited, and properly sampled but does not satisfy the mixing condition.

Under (1)–(3), the bispectrum over the OT is indeed not necessarily zero. Under (4), the variances of the sample estimates are unknown and may be infinite. Thus, the sample bispectrum may appear to be nonzero when the true bispectrum is zero for all bifrequencies. The test can be used to provide inferences about the joint density of the signal if relevant prior knowledge about it is available. For example, if the signal is known to be a random bandlimited stationary process with finite memory, rejection of the test infers the presence of aliasing [5]. If it is known to be random, bandlimited, of finite memory, and properly sampled, the assumption of nonstationarity is rejected.

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