Ocean Acoustic Field Matching, Normal Mode Filtering and Non-Gaussian Sources

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(Invited Paper)

***Abstract*— Location of a submerged object is an important problem in ocean acoustics. Classical interest in statistical sonar signal processing and contemporary interest in acoustic field matching are clear indications of the importance attached to this location problem. This paper produces two qualitative results concerning this quantitative subject. First, this paper integrates normal-mode field predictions with statistical signature analysis by constructing a boundary-value problem in the acoustic waveguide. This construction produces this new result: the normal-mode filter is the unique acoustic pre-processor which does not confound deterministic waveguide correlation structure with stochastic source covariance structure. Second, this paper investigates the origin of deterministic, Gaussian, and non-Gaussian source signatures by associating physical parameters with the classical Lindeberg central limit conditions. This construction produces this new result: There are important objects that are not adequately represented either by infinitesimal points or by infinite surfaces. If receiver resolution is inadequate to resolve source complexity, these objects will exhibit a non-Gaussian acoustic signature via an entirely linear progression from internal excitation, to source radiation, through waveguide propagation, and finally to reception.**

I. Introduction

Location of a submerged object is an important problem in ocean acoustics [1]–[8]. Contemporary interest in acoustic field matching [7]–[17] is a clear indication of the importance attached to this problem. For a more exhaustive bibliography, see [18]. This paper is a qualitative discussion of this quantitative subject. Accordingly, this paper makes three qualitative points.

(1.1) Field matching is not the complete solution of the object location problem; statistical analysis of a stochastic source is required.

(1.2) The normal-mode solution is the unique waveguide basis set which exhibits both quantitative parsimony and qualitative rigor; therefore, the normal-mode filter is the unique acoustic preprocessor which does not confound deterministic waveguide correlation structure with stochastic source covariance structure.

(1.3) There are important objects which are not adequately represented either by infinitesimal points or by infinite surfaces. If receiver resolution is inadequate to resolve source complexity, these objects will exhibit a non-Gaussian acoustic signature via an entirely linear progression from internal excitation, to source radiation, through waveguide propagation, and finally to reception.

The first point is well known but often overlooked. The last two points are new results. Exposition of these points proceeds as follows. Section II integrates normal-mode field prediction methods with stochastic signature analysis methods by constructing a boundary-value problem in the acoustic waveguide. Hilbert space techniques, which are implicit in both the waveguide physics and the signature signal processing, are the unifying concept. This construction demonstrates that matched field processing employs waveguide physics to full advantage but leaves stochastic expansion coefficients requiring statistical analysis. Section III examines the stochastic character of the normal-mode expansion coefficients produced by matched field processing. The physical origins of Gaussian and non-Gaussian statistical fluctuations in the source signature are considered. In particular, it is shown that nonlinear source mechanisms are not required to produce a non-Gaussian signature. Section IV closes the paper with a brief discussion of statistical signature analysis issues.

II. Wave Operator Formulation of the Object Location Problem

Suppose it is desired to detect, locate, and classify an acoustic source residing in an oceanographic waveguide. The desired solution involves some combination of source, signal, propagation, interference, and measurement factors. Since each factor arises from physical entities, the natural approach is to construct a waveguide boundary-value problem from the well-known conservation equations imposed by physical laws. The various conservation equations, initial conditions, boundary values, and source functions are combined with a thermodynamic equation of state to form a set of governing partial differential equations [19], [20]. The coupled partial differential equations are collectively termed a waveguide operator W [21], [22]. For acoustic pressure $p(\vec{x}, t)$ in the fluid interior,

$$W[p(\vec{x}_i, t)] = s(\vec{x}_i, t), \tag{2.1}$$

where

$W[\cdot] \equiv \nabla \cdot c(\vec{x})\nabla(\cdot) - \partial^2(\cdot)/\partial t^2$,
$\vec{x}_i = (x_1, x_2, x_3)$ is the spatial coordinate in the fluid interior V,
$s(\vec{x}_i, t)$ is the signature radiated from the object, and
$c(\vec{x}_i)$ is the fluid sound speed.

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And for acoustic pressure $p(\vec{x}, t)$ on the boundaries,

$$W[p(\vec{x}_b, t)] = f(\vec{x}_b, t) \tag{2.2}$$

where

$W[\cdot] \equiv \alpha(\cdot)|_B + \beta\vec{n} \cdot \nabla(\cdot)|_B$,
$\vec{x}_b = (x_1, x_2, x_3)$ is the spatial coordinate on the boundary surface B,
$f(\vec{x}_b, t)$ is the acoustic constraint on the boundary surface B,
$\vec{n}$ is a vector normal to the surface B, and
α and β are physical parameters.

In underwater acoustics, typical boundary constraints are a pressure release condition on an ocean surface with time-varying roughness, and a continuity condition (producing either partial or total reflection) on an ocean bottom with time-invariant roughness.

For the linear wave operator in (2.1) and (2.2), the normal-mode method obtains pressure field solutions via an infinite series expansion of wave operator eigenfunctions. That is,

$$p(\vec{x}, t) = \sum_{\vec{m}, n} C_{\vec{m}, n} \Psi_{\vec{m}, n}(k_{\vec{m}, n}, \vec{x}, \omega_n, t) \tag{2.3}$$

where

n and $\vec{m} = (m_1, m_2, m_3)$ are normal-mode indexes,
$k_{\vec{m}, n}$ and ω_n are eigenvalues, and
$\Psi_{\vec{m}, n}(\cdot)$ are eigenfunctions defined by

$$\begin{aligned} \langle W[\Psi_{\vec{m}, n}], \Psi_{\vec{m}', n'} \rangle &= \lambda_{\vec{m}, n} \langle \Psi_{\vec{m}, n}, \Psi_{\vec{m}', n'} \rangle \\ &= \lambda_{\vec{m}, n} \delta(\vec{m} - \vec{m}', n - n') \end{aligned} \tag{2.4}$$

with

$$|\lambda_{\vec{m}, n}| = \left(\frac{\omega_n}{c(\vec{x})} \right)^2 = |k_{\vec{m}, n}|^2 = k^2_{m_1, n} + k^2_{m_2, n} + k^2_{m_3, n}$$

(see [23]) and the symbol $\langle \cdot, \cdot \rangle$ is the inner product.

The expansion coefficients are determined from the source signature by

$$C_{\vec{m}, n} = \frac{1}{d_{\vec{m}, n}} \langle s(\vec{x}, t), \Psi_{\vec{m}, n}(k_{\vec{m}, n}, \vec{x}, \omega_n, t) \rangle \tag{2.5}$$

where

$$d_{\vec{m}, n} = \langle \Psi_{\vec{m}, n}, \Psi_{\vec{m}, n} \rangle = \|\Psi_{\vec{m}, n}\|_2^2, \text{ and } \|s(\vec{x}, t)\|_2^2 < \infty.$$

For waveguides exhibiting simple symmetries, standard textbook methods [24], [25], commonly separation of variables or integration of a Green's function, may be employed to construct closed form normal-mode solutions [26], [27]. For realistic oceanography, numerical approximation of these eigenfunctions and associated eigenvalues is usually required.

In principle, (2.1)–(2.5) are the complete solution of the ocean acoustics problem. In practice, these equations mark the middle, not the end, of the solution. *A priori*, the waveguide is never really known with the certainty implied by setting down the symbol $\Psi_{\vec{m}, n}(\cdot)$. *A fortiori*, $s(\vec{x}, t)$ is unknown. This is, after all, remote sensing. The art of employing *a priori* information and stochastic assumptions at the receiver to extract useful, but inevitably partial, information from the wave operator solution is the essence of statistical signal processing. Mathematical physics defines and elucidates the problem; stochastic signal processing confines and completes it. Thus we have made point (1.1): *Field matching is not the complete solution of the object location problem; statistical analysis of a stochastic source is required.*

The art here is in the selection of *a priori* information and the choice of stochastic assumptions. In the forward waveguide problem, detailed knowledge of the source is part of the *a priori* information employed to determine acoustic pressure at remote field points; hence detailed source characterization is the beginning of the forward problem. In the object location problem, an inverse waveguide solution is employed to obtained detailed knowledge of the source associated with each potential object location; hence detailed estimates of source characteristics represent the conclusion of the waveguide solution in the object location problem.

Therefore, the inverse waveguide solution assumes that the remote object is a stochastic source $s(\vec{x}_s, t_s)$ located at position $\vec{x}_s$ and radiating at time t_s. The physical origin and expected statistical character of $s(\vec{x}_s, t_s)$ are the subject of Section III. For the present, we seek to obtain partial knowledge of $s(\vec{x}_s, t_s)$ by relaxing the strictures of W while avoiding gross violation of the physical principles. Specifically, we relax the precision knowledge of $s(\vec{x}_s, t_s)$ implied by (2.3.). In its stead, we require $s(\vec{x}_s, t_s)$ to satisfy a subset of the conservation equations used to construct W. In particular, an energy conservation requirement is obvious. Boundary smoothness constraints [22] are less obvious. Thus $s(\vec{x}_s, t_s)$ is assumed to be a stochastically continuous random field in continuous space and time, such that $E\|s(\vec{x}_s, t_s)\|_2^2 < \infty$, where $E[\cdot]$ denotes the stochastic mean or expected value.

Notice that the inner product plays a crucial role both in defining the stochastic signal processing problem in the previous paragraph and in defining the normal-mode solution. This is no coincidence. Energy conservation is equivalent to the assertion that all possible solutions to the ocean acoustics problem reside in an implicitly defined space of square-integrable functions. This space of square-integrable functions is variously termed signal space in signal processing vernacular [28], Hilbert space in mathematics parlance [29], or simply an inner produce space. Any countable collection of orthonormal expansion functions that spans the implicit Hilbert space (termed a basis set) is, therefore, a mathematically legitimate collection of expansion functions for representing the constituent factors in the ocean acoustics problem. A vast number of such basis sets exist. The trigonometric Fourier series, Bessel functions, and Legendre polynomials are perhaps the most familiar (and most commonly employed) examples. Of all the basis sets which exist on the inner product space defined by conservation of waveguide energy, the set of eigenfunctions are the *unique* basis set that diagonalizes the waveguide operator, W, in the sense of (2.4.).

The signal processing problem that completes the acoustics problem regards the waveguide as a filter which delays, distorts, diminishes, and when fortune smiles, focuses the object signature. Detailed oceanographic information, in the form of a normal-mode solution, is provided to the signal processing problem as *a priori* information at the receiver. Two standard assumptions are worth noting here. First, normal-mode spatial dependence is presumed separable from normal-mode time dependence in this fashion:

$$\Psi_{\vec{m},n}(k_{\vec{m},n}, \vec{x}, \omega_n, t) = \Phi_{\vec{m}}(k_{\vec{m},n}, \vec{x})e^{j\omega_n t}. \tag{2.6}$$

This represents the assumption that temporal waveguide fluctuations are quite slow or are stationary relative to other object location factors, such as transient source excitation (whether active pulses or passive transients), source-receiver relative motion, or cylindrical spreading in active reverberation. Second, the base factor $\Phi_{\vec{m}}(\cdot)$ is presumed deterministic. Stochastic perturbation factors can, of course, be introduced but are not considered here.

The normal-mode solution is intrinsically a linear waveguide approximation. Therefore, the desired inverse waveguide solution is a linear filter. Of all such filters, the choice which maximizes output signal energy at the reception position arising from the object position while minimizing interference arising from other locations is the waveguide matched filter. This waveguide matched filter is derived by the well-known Schwarz inequality manipulation [28], [29] as follows.

Let $p_s(\vec{x}_o, t_o)$ be the sound pressure level observed at location $\vec{x}_o$ and time t_o arising from a source $s(\vec{x}_s, t_s)$ radiating at location $\vec{x}_s$ and time t_s. For reasons that will be presented in more detail in Section III, there are important circumstances where the source and the receiver are not well represented by infinitesimal points; therefore, define their respective locations $\vec{x}_s$ and $\vec{x}_o$ as:

$$\vec{x}_s = \frac{\int_{\text{source}} \vec{x}E|s(\vec{x}, t)|^2 \, d(\vec{x}, t)}{\int_{\text{source}} E|s(\vec{x}, t)|^2 \, d(\vec{x}, t)} \text{ and}$$
$$\vec{x}_o = \frac{\int_{\text{receiver}} \vec{x}E|p(\vec{x}, t)|^2 \, d(\vec{x}, t)}{\int_{\text{receiver}} E|p(\vec{x}, t)|^2 \, d(\vec{x}, t)} \tag{2.7}$$

where the integrals are taken over distinct waveguide volumes for the duration of the signal. Since the critical connection between the waveguide problem and the signal processing problem is the inner product space imposed by conservation of energy, $\vec{x}_s$ is the energy weighted centroid of the effective resolution cell defined in Section III and $\vec{x}_o$ is the energy weighted centroid of the effective phase center of the receiver. For our purposes, these quantities are well-defined provided the respective fluid volumes are disjoint. t_s and t_o are similarly defined as the respective centroids of the effective durations of the signal transmission and observer reception. $s(\vec{x}_s, t_s)$ is a statistically fluctuating cell represented in (distinct) transmission and/or reception intervals by (distinct) $C_{\vec{m},n}$ ensembles. In this representation, $s(\vec{x}_s, t_s)$ is a volume source even if the receiver is unable to resolve this source volume from neighboring volumes.

Now we seek a linear wavenumber-frequency transfer function between $(\vec{x}_s, t_s)$ and $(\vec{x}_o, t_o)$ that maximizes the signal energy at the observation location due to the source at the object location. That is, determine H, with $\|H\|_2^2 < \infty$, so that $\|p_s\|_2^2 = \|E\{S\}H\|_2^2$ is maximum. In the preceding equation, S is the wavenumber-frequency representation of $s(\vec{x}_s, t_o)$ via (2.5) whereby $S_{\vec{m},n}(k_{\vec{m}}, \vec{x}_s, \omega_n, t_s) = C_{\vec{m},n}\Phi_{\vec{m}}(k_{\vec{m},n}, \vec{x}_s)e^{j\omega_n t_s}$. By the Schwarz inequality, $\|HS\|_2^2 \leq \|H\|_2^2\|E\{S\}\|_2^2$ with equality if and only if $H = AE\{S^*\}$, where A is an arbitrary constant. Therefore, $H_{\vec{m},n}(k_{\vec{m}}, \vec{x}, \omega_n, t) = \Phi^*_{\vec{m}}(k_{\vec{m},n}, \vec{x})e^{-j\omega_n t}$, where the arbitrary constants are $A_{\vec{m},n} = 1/C^*_{\vec{m},n}$. If h is the space-time transform of H, then $p_s(\vec{x}_o, t_o) = \langle E\{s(x_s, t_s)\}, h(x_o, t_o)\rangle$. The equivalent wavenumber-frequency implementation of the waveguide matched filter is then

$$p_s(\vec{x}_o, t_o) = \sum_{\vec{m},n} e^{-j(\omega_n t_s)}\Phi^*_{\vec{m}}(k_{\vec{m},n}\vec{x}_s) \cdot E\{C_{\vec{m},n}\}\Phi_{\vec{m}}(k_{\vec{m},n}, \vec{x}_o)e^{j(\omega_n t_o)} \tag{2.8}$$

Thus statistical $C_{\vec{m},n}$ ensembles represent the signal processing closure of the waveguide physics problem. Detailed knowledge of $C_{\vec{m},n}$ combined with $\Phi_{\vec{m}}(k_{\vec{m},n}, \vec{x}_s)e^{j(\omega_n t_s)}$ in (2.8) provide $p_s(\vec{x}_o, t_o)$ ensemble predictions. However, as previously argued, detailed knowledge of $C_{\vec{m},n}$ represents the end of the remote location problem not the middle. That is, $s(\vec{x}_s, t_s)$ is unknown, $p_s(\vec{x}_o, t_o)$ is observed, and $C_{\vec{m},n}$ is to be determined. We proceed by modifying (2.1) for the stochastic model:

$$W[E\{p(\vec{x}, t)\}] = E\{s(\vec{x}, t)\}. \tag{2.9}$$

Taking the inner product,

$$\langle W[E\{p(\vec{x}, t)\}], \Phi_{\vec{m}}(k_{\vec{m},n}, \vec{x})e^{j(\omega_n t)}\rangle = \langle E\{s(\vec{x}, t)\}, \Phi_{\vec{m}}(k_{\vec{m},n}, \vec{x})e^{j(\omega_n t)}\rangle.$$

Exchanging the order of integration and applying the definition of eigenfunctions,

$$E\langle W[p(\vec{x}, t)], \Phi_{\vec{m}}(k_{\vec{m},n}, \vec{x})e^{j(\omega_n t)}\rangle = |k_{\vec{m},n}|^2 E\{C_{\vec{m},n}\}. \tag{2.10}$$

That is, for each observation period, a sample $C_{\vec{m},n}$ is obtained via

$$C_{\vec{m},n} = \frac{1}{|k_{\vec{m},n}|^2}\int_{\text{receiver}} e^{-j(\omega_n t_s)}\Phi^*_{\vec{m}}(k_{\vec{m},n}, \vec{x}_s) \cdot p(\vec{x}, t)\Phi_{\vec{m}}(k_{\vec{m},n}, \vec{x})e^{j(\omega_n t)} \, d(\vec{x}, t). \tag{2.11}$$

For comparison, suppose the signal processing solution employs some other basis functions, say $\varphi_{\vec{m},n}(\vec{k}, \vec{x}, \omega, t)$, which are not eigenfunctions of W. Then (2.10) becomes

$$\begin{aligned} &E\langle W[p(\vec{x}, t)], \varphi_{\vec{m},n}(\vec{k}, \vec{x}, \omega, t)\rangle \\ &\quad = E\{g(W[\cdot], \varphi_{\vec{m},n}(\cdot), C_{\vec{m},n}, k_{\vec{m},n}, \vec{x}_s, \vec{x}_o, \omega_n, t_s, t_o)\} \\ &\quad \sim |k_{\vec{m},n}|^2 E\{C_{\vec{m},n}\} + \text{confounding terms.} \end{aligned} \tag{2.12}$$

Note $g(\cdot)$ is an element of the inner produce space depending in a complicated fashion on the factors indicated.

The corresponding delay-and-sum beamformer amounts to ignoring the waveguide operator W altogether and choosing $\varphi_{\vec{m},n}(\vec{k}, \vec{x}, \omega, t) = \exp[j(k_{\vec{m},n} \cdot \vec{x} + \omega_n t)]$ as basis functions. Then (2.10) becomes the space-time Fourier transform of $p(\vec{x}_o, t_o)$:

$$E\langle p(\vec{x}, t), \exp[j(k_{\vec{m},n} \cdot \vec{x} + \omega_n t)]\rangle = P_o(k_{\vec{m},n}, \omega_n). \quad (2.13)$$

When considered in reverse order, (2.10), (2.12), and (2.13) indicate the successive improvements of first matched-field processing and then matched-mode processing over classical beam formation. Compared to the space-time Fourier transform, the matched field processor in (2.12) improves performance by introducing realistic waveguide modeling as *a priori* information at the receiver. Take careful notice; the improvement in (2.12) over (2.13) arises from the wave operator W, not the basis set $\{\varphi_{\vec{m},n}\}$. In contrast, the subsequent improvement of the matched-mode processor in (2.10) over the standard matched-field processor in (2.12) is precisely in the selection of the eigenfunctions, $\Phi_{\vec{m}}(k_{\vec{m},n}, \vec{x})e^{j(\omega_n t)}$, over all possible alternative basis sets $\varphi_{\vec{m},n}$.

This last conclusion represents at least two distinct points of view. From physics, eigenfunctions are the unique representation which exhibit both quantitative parsimony and qualitative rigor. The absence of confounding terms in (2.10) is a direct result of physics. Alternatively from signal processing, the extra terms on the right hand side of (2.12) confound the deterministic waveguide correlation structure with the statistical covariance structure of the source. If the waveguide also exhibits stochastic character, then this confounding in (2.12) compared to (2.10) is further compounded. Thus we have made point (1.2): *The normal-mode solution is the unique waveguide basis set which exhibits both quantitative parsimony and qualitative rigor; therefore, the normal-mode filter is the unique acoustic preprocessor which does not confound deterministic waveguide correlation structure with stochastic source covariance structure.*

An additional remark is in order here. A normal-mode label on the numerical wave operator approximation procedure is not a fundamental requirement. Suppose one prefers, for example, to trace rays. Then replace the wave operator in (2.1.) and (2.2.) with the appropriate eikonal approximation. Trace rays in the usual manner, taking care to over-sample the acoustic field. Then orthonormalize the ray fans by some standard procedure, say Gram–Schmidt [29]. The resulting rays are now eigenfunctions of the corresponding eikonal operator. Since the preceding theory exploits linear operators and eigenfunctions, not the procedures which approximate them, all preceding conclusions carry forward to the orthonormalized ray solution.

III. Stochastic Source Model

The random expansion coefficients $C_{\vec{m},n}$, derived in Section III, represent the source as observed through an acoustic waveguide. What then is the anticipated statistical character of these coefficients? On the one hand, we may be satisfied with quantitative evidence; it is an empirical fact that ambient ocean noise is Gaussian and shipping noise is non-Gaussian [30], [31]. Alternatively we seek qualitative interpretation; this section investigates the physical origin of deterministic, Gaussian, and non-Gaussian source behavior in ocean acoustics.

Gaussian random variables are observed when the averaging of local characteristics by the physical scenario satisfies the Lindeberg condition and some statistical dependence constrains [32], [33]. In mathematical terms, superposition of certain types of local conditions produce a Gaussian central limit. The most common textbook treatments develop the Lindeberg–Levy central limit theorem to explain this phenomena. The Lindeberg–Levy theorem employs the strongest possible constraint on dependence (i.e., independence) and while employing the weakest possible constraint on variance (i.e., the Lindeberg condition). A Gaussian central limit can be obtained under much milder dependence conditions; relaxation of the Lindeberg condition produces either a non-Gaussian central limit or no limit at all. A brief summary of standard Gaussian and non-Gaussian central limit theorems is provided in the Appendix.

We shall forgo mathematical formality in the remainder of this section in favor of physically intuitive exposition. However informal the language may appear, the reasoning is rigorous.

An engineering paraphrase of the various mathematical conditions alluded to above is that the superposition of physical details converges to a Gaussian central limit when:

(3.1)

i) An effectively infinite number of individual contributors are superposed.
ii) No finite subset of contributors dominate the observed energy.
iii) For any specific contributor and any particular reference dependence, all but a finite number of contributors exhibit less statistical dependence on that specific contributor than the reference dependence.
iv) The probability of tail events must not be too large.

A deterministic signal emerges when one or a few contributors fix the signature or an infinite number of perfectly correlated contributors fix the signature. A sine wave from a projector and specular reflection from a perfect plane are respective examples. Ambient noise from wind driven wave action represents a Gaussian example:

- the number of wave crests and troughs are effectively infinite;
- nearby crests and troughs are correlated; distant crests provide independent but essential contributions;
- large wave height excursions are quite unlikely;

The deterministic and Gaussian sources represent opposite extremes; respectively, a few tell the whole story or an egalitarian all is the only story.

In the absence of nonlinear phenomena, non-Gaussian sources are intermediate to these extremes. Many contributors are important. Distinguishing a few more contributors by increasing measurement resolution (alternatively, distinguishing a few less by reducing resolution) changes the observed signature but does not appreciably simplify the statistical complexity. Dependence between contributors is substantial but imperfect; occasionally a considerable number of contributors will align to produce a signature spike.

Consider long range, passive reception of shipping noise as an example of a non-Gaussian source. Contributions from these four factors:

- salient geometric dimensions of the object $\equiv \vec{L} = (L_1, L_2, L_3,)$ (say the length and draft of the ship together with the tip-to-tip diameter of the propulsor);
- receiver resolution cell dimensions $\equiv \vec{R} = (R_1, R_2, R_3,)$ (following [34], which extends the echo-ranging theory in [35] to passive observation, the smallest volume which can be distinguished from a nearby volume of the same geometry);
- source normal-mode wavenumbers $\equiv \vec{k} = (k_1, k_2, k_3,)$ (the subset of internally excited source modes which satisfy a radiation condition by matching an external waveguide normal-mode); and
- source normal-mode density $\equiv \xi(\omega)$ (following [36] the number of modes (spatial frequencies) arising from the source per unit temporal frequency)

produce the radiated signature. To simplify the discussion, define these bounds on $\vec{L}$, $\vec{R}$, and $\vec{k}$:

$$\begin{aligned} l_{\min} &= \min(L_i) \le \max(L_i) = L_{\max}, \\ r_{\min} &= \min(R_i) \le \max(R_i) = R_{\max}, \text{ and} \\ k_{\min} &= \min(k_i) \le \max(k_i) = K_{\max}. \end{aligned}$$

Now we use these bounds to demonstrate that reasonable combinations of $\vec{L}$, $\vec{R}$, $\vec{k}$, and $\xi(\omega)$ violate the paraphrased Lindeberg conditions i)–iv) for a Gaussian signature in (3.1). Specifically, suppose the following.

(3.2)

a) The resolution size of the ship is small but not infinitesimal, say $0 < L_{\max}/r_{\min} < 1$.
b) The acoustic size of the resolution cell is very large or effectively infinite, say $k_{\min} r_{\min} \gg 1$.
c) The acoustic size of the ship normal-modes occupy a band including unity, say $0 \ll k_{\min} l_{\min} < 1 < K_{\max} L_{\max} \ll \infty$.
d) And the ship modal density, $\xi(\omega)$, in the acoustic band between $k_{\min} l_{\min}$ and $K_{\max} L_{\max}$ is large but not infinite, is discrete but not sparse.

For concreteness, one may consider that (3.2) describes propulsor induced vibrations radiating from the ribbed expanse of the ship's hull from the keel to the water-line. Specifically, a) and b) state that the local signature details are all superposed in one receiver resolution cell; however, c) asserts that the ship is emphatically not a simple source. $k_{\min} r_{\min} \gg 1$, resolution cell diffraction is insignificant; $L_{\max}/r_{\min} < 1$, the receiver sees a point. But $\vec{k} \cdot \vec{L} \gg 0$, the radiating hull is not a point; and $\vec{k} \cdot \vec{L} \ll \infty$, the signature is not specular radiation.

c) and d) state that small changes in local details produce significant signatures variations. Sections of the ship where the rib spacings differ or the hull changes concavity will produce distinct space-time signatures. Small perturbations, say hull rocking due to wave action, yield significant signature fluctuations at the reception position. $\vec{k} \cdot \vec{L} \gg 0$ and $\xi(\omega) \ll \infty$, many distinct contributors tell the story; $\vec{k} \cdot \vec{L} \ll \infty$ and $\xi(\omega) \gg 1$, excluding a few contributors does not simplify the plot; $\vec{k} \cdot \vec{L} \sim 1$, there is an occasional thunder clap.

Finally, there is substantial but imperfect stochastic dependence in the radiated modes. Strong dependence arises because the propulsor-drivetrain combination is the fundamental driver for all radiating modes; this coherence is retained in distinct modes. However, imperfections in this dependence are introduced by complex foundations, flexible drive joints, and near but not exact structural periodicity. That is, the signature components radiating from forward sections, aft sections, and the hull bottom itself exhibit significant coherence due to common excitation but also exhibit non-negligible distinctions due to physical nonuniformity.

Thus shipping noise satisfying (3.2) violates each of the paraphrased Lindeberg conditions (3.1). The crucial conclusion is that a number of important objects exhibit non-Gaussian acoustic signatures. The critical requirements are many distinct contributors with substantial but imperfect statistical dependence where the distinction between contributors cannot be adequately resolved. Thus, we have made point (1.3): *There are important objects that are not adequately represented either by infinitesimal points or by infinite surfaces. If receiver resolution is inadequate to resolve source complexity, these objects will exhibit a non-Gaussian acoustic signature via an entirely linear progression from internal excitation, to source radiation, through waveguide propagation, and finally to reception.*

IV. Statistical Analysis of the Matched Field Expansion Coefficients

The matched-mode waveguide filter in (2.10) produces random expansion coefficients $C_{\vec{m},n}$ which describe the stochastic source $s(\vec{x}_s, t_s)$ located at position $\vec{x}_s$ and radiating at time t_s as measured by $p_s(\vec{x}_o, t_o)$ at the observation point. Statistical analysis of these ensembles is required to complete the object location problem. For real-valued ensembles, the multivariate joint probability distribution function, $P\{C_{\vec{m},n} \le r_{\vec{m},n}, C_{\vec{m}',n'} \le r_{\vec{m}',n'}, \cdots\}$ for all real $r_{\vec{m},n}$ and all discrete indexes $\vec{m}$ and n for each location, is in principle the complete solution of the statistical signal processing problem. That is, for each source parameter of interest, employ P to form the associated parametric likelihood function; then choose the parameter value which maximizes this likelihood function. However, just as in the practice of waveguide problems, detailed knowledge of P indicates the end of the signal processing solution not the beginning. That is, $P\{C_{\vec{m},n} \le r_{\vec{m},n}, C_{\vec{m}',n'} \le r_{\vec{m}',n'}, \cdots\}$ is unknown, sample ensembles of $C_{\vec{m},n}$ are observed, and some statistical estimate of location is to be constructed.

This is another juncture where stochastic assumptions are introduced at the receiver in order to confine and complete the solution. An appropriate statistical source model is selected by choosing the sort of object to be located. That is, *a priori* information is employed to select a source model, whether it be deterministic, Gaussian, or non-Gaussian, based perhaps on the reasoning presented in Section III. An associated interference model is also selected.

Having selected models for source and interference, suitable attributes of the joint probability distribution function, which can be estimated directly from the observations, are chosen. Estimating statistical cumulants [37], [38] is a natural choice. For the special case of a real-valued random process $X(t, \Omega)$ where observations are available in the form of scalar-valued ensembles and the associated probability distribution function is $P\{X(t, \Omega) \leq r\}$, the statistical cumulants κ are defined by:

$$\ln E[e^{j(sX)}] = \ln \int_{r=-\infty}^{r=\infty} e^{j(sX)}\, dP\{X(t, \Omega) \leq r\} = \kappa_1 s + \frac{\kappa_2 s^2}{2!} + \frac{\kappa_3 s^3}{3!} + \cdots \frac{\kappa_n s^n}{n!} + o(|s^{n+1}|). \tag{4.1}$$

That is, the nth-order cumulant, κ_n, is the nth coefficient in the Maclaurin series expansion of the left side of (4.1). For $X(t, \Omega)$, cumulants 1 and 2 are the familiar statistical parameters mean and variance. If $X(t, \Omega)$ is normalized to unit variance, then κ_3 is the skew and κ_4 is the kurtosis. For the random field $C(\vec{m}, n, \Omega)$, where observations are available in the form of vector-valued ensembles of normal-mode coefficients, $C_{\vec{m}, n}$, the definition and interpretation of κ is conceptually similar although the analytical details are decidedly more complex.

If deterministic or Gaussian models are reasonable source approximations and the additive interference is Gaussian, then maximum likelihood estimation of the second-order cumulants is the complete statistical story. If the normal-mode coefficients, $C_{\vec{m}, n}$, are stationary, this amounts to classical stationary power spectral analysis [39], [40]. For instance, the Hinich–Sullivan procedure [7] employs matched-mode filtering by a vertical array in a range-invariant but vertically inhomogeneous waveguide followed by maximum likelihood location estimation of a deterministic source in Gaussian interference. For a stationary Gaussian source, the approach proposed here would be an extension of the second-order maximum likelihood procedure in [5] to matched-mode processing. If the received signal is periodic or almost periodic, then analysis of the mode coefficient ensembles would require a nonstationary cumulant method, say spectral correlation [41], [42].

It should be pointed out that for the Gaussian composite signal, the canonical maximum likelihood method may be readily employed in the form of a linear least squares error procedure [1], [2]. However, for this Gaussian case there are a number of other estimation procedures in contemporary vogue, albeit with reduced resolution and accuracy. For instance, the Capon estimate of power emanating from a particular location is employed in the Baggeroer–Kuperman–Schmidt matched field procedure [12] in lieu of the maximum likelihood estimate of location. See Brillinger [6] for a detailed comparison of the Capon procedure with the canonical maximum likelihood estimator.

For a non-Gaussian source, third (and perhaps higher order) cumulant estimation should be considered. For non-Gaussian sources in an acoustic waveguide, the present approach represents a combination and extension of the Hinich–Sullivan normal-mode procedure with the Hinich–Wilson polyspectral estimation methods [43], [44]. This extension represents the real possibility of distinguishing between multipath signals from a single source and single path arrivals from multiple sources.

For higher-order cumulants the question of the preferred estimator remains open. The maximum likelihood estimator in this case is a nonlinear function of the observations, a decidedly unpleasant circumstance. Linear approximations of the nonlinear maximum likelihood estimator are at most locally optimal. However, as pointed out in [45], even a suboptimal estimate of higher order cumulants is a significant improvement over second-order cumulant analysis when a non-Gaussian source is present.

V. Appendix

This appendix summarizes well-known central limit theorems. Detailed development and proof may be found in standard references. In this discussion, the notation is taken from [32] and the examples from [33].

Consider a sequence of random variables $\{X_k,\, k \geq 1\}$ with $E[X_k] = 0$, $E[X_k^2] = \sigma_k^2$, $s_n^2 = \sum_1^n \sigma_k^2$, and probability distributions $\{P_k,\, k \geq 1\}$.

Definition A-1: Lindeberg Condition: The sequence of random variables $\{X_k,\, k \geq 1\}$ is said to satisfy the Lindeberg condition if $\sigma_k^2 < \infty$ for all $k \geq 1$, $s_n^2 > 0$ for some $n \geq 1$, and $\lim_{n\to\infty} (1/s_n^2) \sum_1^n \int_{|X_k| > \epsilon s_k} X_k^2\, dP_k = 0$ for all $\epsilon > 0$.

Definition A-2: Gaussian Central Limit: For $S_n = \sum_1^n X_k$, the normalized sequence of partial sums $\{S_n/s_n,\, n \geq 1\}$ with the associated sequence of probability distributions $\{P_n,\, n \geq 1\}$ is said to converge to a Gaussian central limit if the $\lim_{n\to\infty} P_n = N(0, 1)$.

Provided the Lindeberg condition is satisfied, an interesting range of statistical dependence constraints support a Gaussian central limit:

- complete independence (most theoretically tractable),
- finite or M-dependence (easiest practical dependence model to implement in real systems),
- α-mixing or asymptotic independence (arises naturally in certain Markov chains).
- Martingale differences (mildest dependence constraint in this list).

If the Lindeberg condition is explicitly violated, the authors are not aware of an alternate constraint whereby a Gaussian central limit can be recovered. In these cases, the normalized sequence of partial sums, $\{S_n/s_n,\, n \geq 1\}$, either converges to a non-Gaussian central limit or simply fails to converge to any limit. Non-Gaussian central limits are more common than one might expect.

For example, sequences of partial sums of independent Cauchy, exponential, or gamma distributed random variables converge to Cauchy, exponential, and gamma central limits, respectively. These are examples of the class of so called stable distributions. The class of stable distributions is, in fact, quite large. For $0 < \alpha \leq 2$, the probability distribution P_α associated with the characteristic function $\varphi_\alpha(t) = e^{-|t|^\alpha}$ is a stable distribution.

A more general class of distributions which support central limits is the class of infinitely divisible distributions. Stable distributions are a special case of infinitely divisible distributions. The well-known convergence of certain sums of binomial random variables to a Poisson central limit is an example of an infinitely divisible distribution which is not a stable distribution.

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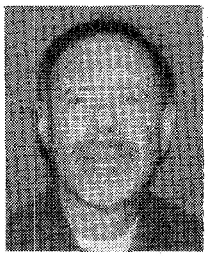

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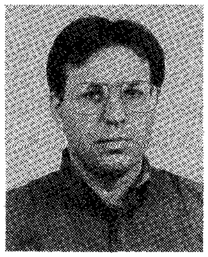

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