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A general probabilistic spatial theory of elections*

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3. CITIZEN. We have been called so of many, not that our heads are some brown, some black, some auburn, some bald, but that our wits are so diversely colored. And truly I think if all our wits were to issue out of one skill, they would fly east, west, north, south, and their consent of one direct way should be at once to all points o' the compass. (Shakespeare, Coriolanus II.iii. 19–26)

Abstract. In this paper, we construct a general probabilistic spatial theory of elections and examine sufficient conditions for equilibrium in two-candidate contests with expected vote-maximizing candidates. Given strict concavity of the candidate objective function, a unique equilibrium exists and the candidates adopt the same set of policy positions. Prospective uncertainty, reduced policy salience, degree of concavity of voter utility functions, some degree of centrality in the feasible set of policy locations, and restrictions on the dimensionality of the policy space are all stabilizing factors in two-candidate elections.

1. Introduction

The spatial theory of elections is based on the premise that the policy positions of voters and candidates can be represented by points in an issue space and that a voter's evaluation of a candidate's policy positions is measured by the distance between voter and candidate in this space. If candidates have spatial mobility, the purpose of the theory is to predict where each candidate will locate in the issue space if he wishes to win the election.

Whether or not it is possible to assess the policy positions of voters and candidates, there are always unobservable variables that affect voter choice. Furthermore, policy positions are always measured with error. These considerations suggest the need for a behaviorally reasonable theory of voting which incorporates the essential uncertainty that candidates have about voter choice and that voters have about candidate positions.

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The voter's uncertainty about the candidates may arise from several sources. Candidate policy positions may be imperfectly perceived or may be perceived as a random variable. Uncertainty about new issues and future events may also complicate the voter's decision problem. A voter who is future-oriented must face this inescapable uncertainty, even if he is confident that he knows the candidates' positions on current policy issues.

A candidate, on the other hand, faces the uncertainty of never knowing all the factors that affect citizens' vote decisions. Even when voters are rational, informed, and have clearly defined views on policy issues, the candidate still cannot be certain about how the votes will be cast. In addition, the data he possesses are likely to contain a large amount of error.

In this paper, we will construct a behaviorally reasonable theory of two-candidate competition designed to reflect electoral uncertainty by both voters and candidates. Each candidate seeks to maximize his expected vote, which is a function of the measurable difference in policy utilities between the candidates as well as the distribution of an unobserved variable. This variable may represent the difference in nonpolicy attributes between two candidates, or any type of uncertainty which varies across voters as a function of the measurable policy difference between the candidates and which is distributed independently of this difference. Our underlying assumption about voter behavior is a generalization of classical deterministic theory, which assumes that this random variable is discrete with probability mass equal to one at the point where the policy utility difference between the candidates is zero.

In contrast with earlier work (e.g., Hinich, Ledyard and Ordeshook, 1973), we deliberately choose to analyze the problem at the level of the entire electorate, rather than build an explicit citizen-level theory of probabilistic choice. Our reason for proceeding this way is to avoid tying our results to any specific rationalization of the candidate objective function. In Enelow and Hinich (1982, 1984b) we model the random element in the voter's calculus as the difference in candidate abilities that are independent of candidate policy positions. This is one, but only one, avenue that may be used to justify the theory we present.

After constructing our theory of candidate competition, we analyze a sufficient condition for a pure strategy equilibrium in a multidimensional policy space in order to understand the conditions under which it will be satisfied. To aid in this understanding, we focus on the case of a single policy dimension with quadratic policy preferences and a normally distributed random element. We conclude by discussing the informational requirements of the probabilistic theory in comparison with those of deterministic spatial theory.

We see our results as relevant to the question raised by Tullock (1981), who asked: Why so much stability? The goal of positive analysis should be to explain stability and instability with the same theory. Accordingly, the question we aks is: Which factors are linked with stability, and which with instability? We show that (1) the magnitude of the variance of the random term in our theory provides part of the answer to this question. Given sufficient dispersion of this random element in the voting population, stability in candidate competition is ensured; given insufficient dispersion, instability may result. (2) Reducing the salience of policies to voters and (3) limiting the feasible set of candidate policy locations also help produce stability in two-candidate elections. The opposite of these conditions makes instability more likely. (4) The degree of concavity in voter utility functions is also linked to stability in two-candidate competition. Linear utility functions make candidate stability less likely. (5) The dimensionality of the policy space is also important. As the number of policy dimensions increase, the sufficient condition for equilibrium becomes harder to satisfy.

In a well-known article, Riker (1980: 443–444) states that 'disequilibrium, ..., is the characteristic feature of politics.' Riker's opinion is that of the institutionalist school, which sees stability in collective choice as an artifact of the way in which institutions narrow down the feasible set of policies. Stability, from this standpoint, is spurious, since it will be upset as soon as the institution which creates it is changed.

Narrowing the feasible set of policies can induce equilibrium, but our explanation of electoral stability stresses the inclusion of unobservable but rational factors in human decision making, *not* the deliberate exclusion of feasible alternatives. Elections take place between individuals who seek the trust of electors. These electors are not choosing between ballot propositions, but, instead, must seek to determine which candidate will most effectively lead them. Uncertainty about what these candidates represent and how they will perform if elected, clouds the choice process and leads to a certain degree of unpredictability in the minds of voters and candidates. It is this unpredictability, much less present in pure policy voting, which our theory captures and which we see as important to understanding the conditions conducive to stability in electoral competition.

2. A general probabilistic theory of two-candidate elections

The theory we construct is standard in several respects. Two candidates, R and T, compete for votes in a multidimensional policy space. Each candidate wishes to maximize his expected vote. As shown in Aranson, Hinich and Ordeshook (1974), for two-candidate elections with constant turnout, this objective function is equivalent to either maximizing expected plurality or maximizing the candidate's proportion of the expected vote. In addition, as the number of voters approaches infinity, maximizing the expected vote is equivalent to maximizing the probability of winning the election.

Let u(r,i) represent voter i's policy utility for candidate R and u(t,i) voter i's policy utility for candidate T, where $r = (r_1, \ldots, r_m)$ and $t = (t_1, \ldots, t_m)$ are m-dimensional vectors of policy positions in a compact,[1] convex subset X of m-dimensional Euclidean space, and i is a member of a population of voters. For notational simplicity, the voter index will henceforth be dropped and the utility function written u(r) or u(t). The utility function u is assumed to be continuous and twice differentiable.

Define F as a distribution function with density f for the difference in R and T's attributes that are independent of r and t. The expected vote share for R is, then,

$$V(r) = E[F(u(r) - u(t))], \tag{1}$$

where E denotes expectation with respect to some probability measure on the index set i (for voter i in the population) and F(u(r) − u(t)) is the conditional probability that all voters with the same policy utility difference between R and T vote for candidate R.[2]. F is assumed to be continuous and twice differentiable. Candidate T's objective function is V(t) = 1 − V(r).

It is easier to view F as a unidimensional distribution function, but our results also hold if F is a joint distribution function of independent random variables. For simplicity, however, we will assume that F is unidimensional.

The following additional assumption is sufficient for the existence of equilibrium in this two-candidate zero-sum game. Before stating it, however, we need to define more terms. Define u_j as the first partial derivative of u with respect to r_j and u_{jk} as the second cross partial derivative of u with respect to r_j and r_k. Let u_r be the m-dimensional column vector of first partial derivatives of u, $u_r = (u_1, \ldots, u_m)$. The m × m matrix, denoted H_u, of the u_{jk} elements is called the Hessian of u with respect to r. Finally, if x is a column vector, let x^T denote the transpose of x.

Condition 1. If x is any nonzero m-dimensional column vector, then

$$-E\,[f(d)x^T H_u x] \geq E\,[f'(d)[x^T u_r]^2] \text{ for all r,t in X} \tag{2}$$

where d = u(r) − u(t), and the expectation is taken with respect to the probability measure defined on the voting population.

To better understand Condition 1, which is necessary and sufficient for concavity of V(r), assume that f(d) = c exp(−g(d)) where g is a convex function for all d such that f(d) > 0, and c > 0 is a scale factor. Such densities are known as PF_2 densities, and include the normal, exponential, gamma, beta, logit, noncentral t-densities and nearly every class of continuous density used in statistics or the modelling literature. For example, the standard normal density is $f(x) = c \exp(-x^2/2)$, where c is the normalizing constant, so $g = x^2/2$, which is a convex function of x.

Condition 1 is satisfied if $-f(d)x^T H_u x \geq f'(d)[x^T u_r]^2$ for each voter i. If $f(d) = \exp(-g(d))$, then $f'(d) = -g'(d)f(d)$ and, if $x^T u_r \neq 0$, we can rewrite this condition as

$$-x^T H_u x/[x^T u_r]^2 \geq -g'(d) \text{ for all r,t in X and for all i} \tag{3}$$

If u is jointly concave in all its variables, H_u is negative semi-definite and the left-hand side of (3) is nonnegative. We can see, therefore, that concavity of the voter utility functions (also known as risk aversion) makes it easier to satisfy Condition 1.

Because X is compact and convex, there will always exist an upper bound to $-g'(d) = f'(d)/f(d)$. Let b denote the least upper bound of $-g'(d)$. One way to keep b close to zero is for $f'(d)$ to be gradual. As F(d) flattens out (i.e., the variance of the random variable increases), b approaches zero. For the normal density with zero mean, $-g'(d) = -d/\sigma^2$, so as σ increases, $-g'(d)$ approaches zero. We conclude that if u is concave, increasing the variance of the random variable makes it easier to satisfy the sufficient condition for candidate equilibrium.

If f is a uniform density then $-g'(d) = 0$. If, in addition, u is concave, Condition 1 will *always* be satisfied, as long as $f(d) > 0$, for all d. In this case, the probability of voting for either candidate is linear in the utility difference between the candidates.

Another way of keeping $-g'(d)$ near zero holds for symmetric f even when the variance of the distribution is small. This approach uses the fact that f' will be zero at the mean of d (recall that F is twice differentiable). Thus, as long as d is contained in a small interval around its mean, $-g'(d)$ will be small, regardless of the variance of F. We conclude that if u is concave, limiting the feasible set of candidate policy locations makes it easier to satisfy the sufficient condition for candidate equilibrium.

If the policy space X is one-dimensional,

$$-x^T H_u x/[x^T u_r]^2 = -u''(r)/[u'(r)]^2 \tag{4}$$

For concave u, the right-hand side of (4) is similar to the Pratt-Arrow measure of absolute risk aversion (Arrow, 1970), $R_A(x) = -u''(x)/u'(x)$, but has the square of the denominator. Lindbeck and Weibull (1987) refer to $|u''(x)|/[u'(x)]^2$ as the concavity index of the utility function. We refer to the size of (4) as the degree of concavity of the voter's utility function.

If the policy space is multidimensional, we can better understand when (3) will be fulfilled by establishing a lower bound on the left-hand side of the inequality. Since H_u is symmetric, $-x^T H_u x \geq \lambda x^T x$, where λ is the minimum eigenvalue of $-H_u$. In addition, by the Schwarz inequality, $[x^T u_r]^2 \leq x^T x u_r{}^T u_r$. Thus,

$$-x^T H_u x / [x^T u_r]^2 \geq \lambda / u_r{}^T u_r \quad (5)$$

If u is concave, then $-H_u$ is positive semidefinite and all its eigenvalues are nonnegative.

If the voter's utility is measured by the negative of squared, simple Euclidean distance, we can express the right-hand side of (5) in even simpler form. Specifically, suppose that $u(r) = -(a/2)(r_1-x_1)^2 - \ldots -(a/2)(r_m-x_m)^2$, where $(x_1, \ldots, x_m)$ is the voter's ideal point in m-dimensional Euclidean space, and $a/2 > 0$ is the common salience of the dimensions to the voter. In this case, $-H_u$ is diagonal, $u_{jk} = 0$ for $j \neq k$ and $u_{jj} = -a$ for $j = 1, \ldots, m$. In addition, $u_j = -a(r_j-x_j)$ for $j = 1, \ldots, m$. Since $-u_{jj}$ is the minimum (and maximum) eigenvalue of $-H_u$, it follows that the right-hand side of (5) equals $a/(u_1{}^2 + \ldots + u_m{}^2) = 1/a[(r_1-x_1)^2 + \ldots + (r_m-x_m)^2]$. As m (the dimensionality of the policy space) increases, the lower bound on $-x^T H_u x / [x^T u_r]^2$ decreases, making it harder to satisfy inequality (3). We conclude that increasing the dimensionality of the policy space makes disequilibrium more likely. This finding mirrors Greenberg's (1979) result for deterministic spatial theory.

Another implication of (5), which is most clearly seen for negative quadratic utility, is that Condition 1 is easier to satisfy as (1) the salience of any dimension decreases, and (2) the distance from voter to candidate declines.

It follows from the candidate objective function (1) that if V(r) is concave in r for fixed t, then V(r) is convex in t for fixed r. Given also that X is compact and convex, and that F and u are continuous, we know from Owen (1982: Th. IV.6.2) that an equilibrium exists in pure strategies. Strict concavity of V implies the existence of a unique pure strategy equilibrium. The proof of the following theorem is given in the appendix.

Theorem. V(r) is concave in r if and only if *Condition* 1 holds.

To better understand how this result is obtained, assume a finite population N of voters, each of whom sees a different policy difference between the two candidates. Then,

$$V(r) = E[F(d)] = [F(d_1) + \ldots + F(d_N)]/N \quad (6)$$

As the variance of the random variable increases, F flattens out until it becomes a linear function of d. When this happens, $F(d) = c + ad$, where c,a are constants and $a > 0$. Substituting in (6), we then have

$$V(r) = c + (d_1 + \dots + d_N)\, a/N \tag{7}$$

Since $d = u(r) - u(t)$, if u is concave, then for fixed t the right-hand side of (7) is the sum of N concave functions, which is itself a concave function. The more strictly concave each utility funtion is, the less F has to approach linearity in d for this sum to be concave. In other words, a trade-off exists between the flatness of F and the roundness of the individual utility functions.

The first-order condition for an expected vote maximizing set of policy positions, which can be found in the Appendix, is $E[f(d)u_r] = 0$. If V(r) is strictly concave, a unique pair of equilibrium strategies exists for the two candidates. To prove that these strategies must be the same, let r^* and t^* be these two strategies and assume that $r^* \neq t^*$. Then, because V(r) is strictly concave, R can only do better if T adopts r^* and R sticks with r^*, so R's expected vote in equilibrium is less than F(0). Similarly, R can only do worse if he adopts t^* and T sticks with t^*, so R's expected vote in equilibrium is greater than F(0). Thus, R's expected vote in equilibrium is both greater than and less than F(0), contradicting the assumption that $r^* \neq t^*$. This argument does not depend on the shape of f or u.

If V(r) is linear, there may be multiple equilibria. However, if f is symmetric about $d = 0$, T's objective function is $V(t) = E[1-F(d)] = E[F(-d)]$, which is the same as R's objective function with u(t) in place of u(r). Thus, both candidates will adopt the same position and receive one-half the total vote.

If f is symmetric about $d = 0$ and u is concave, then there is always some location which is a *local* equilibrium for both candidates. Substituting $d = 0$ in Condition 1, $f'(0) = 0$, so the right-hand side of (2) equals 0 and, since H_u is negative semidefinite, the condition always holds.

3. Applications

We will now construct an extended example to provide a better understanding of what we have shown. For the sake of simplicity, we will assume the quadratic utility function given in the preceding section, i.e., $u(r) = -(a/2)(r-x)^2$, where x is the voter's ideal point, both x and r are measured on a one dimensional scale, and $a > 0$. We will also assume that the density of the unobserved random variable is normal with zero mean, i.e., $f(x) = c \exp(-x^2/2\sigma^2)$, where c is the normalizing constant. Finally, we will assume that the voting population is divided into two groups, one with $x = 0$ and the other with $x = 1$. Let N_0 be the fraction of the population with $x = 0$ and N_1 be the remaining fraction with $x = 1$, so that $N_0 + N_1 = 1$.

For V(r) to be concave, we have stated in the preceding section that a sufficient condition is

$$-u''(r)/[u'(r)]^2 \geq -g'[u(r) - u(t)] \text{ for all r,t in X; x} = 0,1 \quad (8)$$

For our example, (8) reduces to

$$1/[a(r-x)]^2 \geq [(r-x)^2 - (t-x)^2]/2\,\sigma^2 \text{ for all r,t in X; x} = 0,1 \quad (9)$$

Let D represent the maximum difference between $(r-x)^2$ and $(t-x)^2$. It makes no sense for either candidate to locate outside [0,1], since he can be defeated by a candidate locating slightly closer to [0,1], so $D \leq 1$. For simplicity, assume that $a = a_0$ for all voters with $x = 0$ and $a = a_1$ for all voters with $x = 1$. Given, in addition, our assumption concerning the distribution of voter ideal points, we can reduce (9) further to

$$1/a_0^2r^2 \geq 1/2\,\sigma^2 \text{ for all x} = 0 \text{ and}$$
$$1/a_1^2(r-1)^2 \geq 1/2\,\sigma^2 \text{ for all x} = 1 \quad (10)$$

so that a sufficient condition for concavity of V(r) becomes

$$\sigma^2 \geq a_0^2r^2/2 \text{ for all x} = 0 \text{ and } \sigma^2 \geq a_1^2(r-1)^2/2 \text{ for all x} = 1 \quad (11)$$

Since r^2 will never exceed 1, and neither will $(r-1)^2$, we can replace (11) with the joint condition

$$\sigma^2 \geq a_0^2/2 \text{ for all x} = 0 \text{ and } \sigma^2 \geq a_1^2/2 \text{ for all x} = 1 \quad (12)$$

In general, reducing the salience of policy dimensions and increasing the variance of the unmeasured difference between R and T are both ways of making this condition easier to satisfy.

To complete our example, the first-order condition for V(r) to be maximized is $E[f(d)u'(r)] = 0$, where $d = u(r) - u(t)$. Since, for our example, $u'(r) = -a(r-x)$ and f is symmetric about 0, the first-order condition will be satisfied when $E[-a(r-x)] = 0$. Assuming a is independent of x, for our two-group example this condition solves to yield the expected vote maximizing solution, $r = E(x) = N_1/(N_0 + N_1) = N_1$, the mean ideal point in the voting population. The objective function for candidate T yields the same first-order condition, so the mean ideal point is an equilibrium for both candidates. If the a weights differ across voters, our mean result will be weighted by these parameters. In a multidimensional policy space, we obtain a similar result.

In simple terms, nonpolicy considerations or other disturbances *indepen-*

dent of the measurable policies of the candidates can cause the candidates to adopt the same policies. The kind of disharmony of views that destabilizes elections when these views concern policies can stabilize elections when these views concern the nonpolicy attributes of the candidates. Sufficient subjectivism among voters is conducive to electoral stability.

A second application of our model is the linear case, $u(r) = ar$ where the salience weight a is the slope of i's utility function. For this case, the necessary and sufficient condition for concavity of V(r) reduces to

$$E[f'(d)a^2] \leq 0 \text{ for all r,t in X} \tag{13}$$

where, as before, $d = u(r) - u(t)$. This inequality will hold if a is independently distributed in the voting population and

$$E[f'(d)] \leq 0 \text{ for all r,t in X} \tag{14}$$

This condition is quite restrictive. To guarantee that (14) holds for *all* possible r and t, f must have nonpositive slope throughout its domain. Otherwise, V(r) will not be concave for all r (and fixed t) in X. As explained before, the roundness of the voter utility functions helps compensate for a certain amount of convexity in F. If utility functions are all linear (i.e., flat), then any convexity in F leads to convexity in V(r). For example, suppose f(d) is normal with mean zero. Choose r,t in X such that $d < 0$ for many voters. Then, V(r) will be convex for r in this region of X. We conclude that linear utility functions are likely to cause disequilibrium in two-candidate elections. This finding is consistent with the results of Feldman and Lee (1987).

4. Discussion

In deterministic spatial theory (e.g., Davis and Hinich, 1966; Plott, 1967; McKelvey, 1976; Schofield, 1978) strong assumptions are required concerning the information that candidates possess about voters. The knife-edge assumption that a voter votes with certainty for the candidate closest to him requires a set of candidates who can measure voter opinion without error. By contrast, we assume that candidates see voter opinion as imperfectly measured, and the causation of voting behavior as imperfectly understood, thus including a random term in their vote calculations.

How much must the candidates know about this random term? Technically, candidate R must be able to calculate his expected vote share $V(r) = E[F(u(r) - u(t))]$, given the distribution function F characterizing his beliefs about how this random term is distributed in the population. Although E and F cannot be reversed on the right-hand side of the expected vote equation, the candidate's informational requirement comes quite close to assuming that the candidate must know only the average policy utility difference between himself and his opponent for each pair of policy positions. This requirement appears at least as reasonable as the assumption that the candidates know voter opinion without error. The scarcity of equilibrium in the deterministic theory implies that the candidates must know far more than the location of some type of median or mean voter.

The results that have flowed from deterministic and probabilistic spatial theories are quite different. Work on the deterministic theory stresses the instability of the electoral process, while work on the probabilistic theory (Hinich, Ledyard and Ordeshook, 1973; Coughlin and Nitzan, 1981; Enelow and Hinich, 1982, 1984a, 1984b; Coughlin, 1986a, 1986b) has arrived at quite opposite conclusions. Given certain conditions on the random element in the voter's calculus, electoral equilibrium exists regardless of the dimensionality of the policy space. Further, the characteristics of this equilibrium are generally attractive, whether from the standpoint of a social welfare function (Coughlin, 1986b; Lindbeck and Weibull, 1987) or in terms of representing a 'golden mean' (Enelow and Hinich, 1984a, 1984b).

It has been pointed out, however, that strong assumptions are usually invoked to reach these optimistic findings for probabilistic voting theory. Coughlin (1986a), for example, relies on a binomial logit model, or a concave, binary strict utility model (Coughlin and Nitzan, 1981; Coughlin, 1986b); while Enelow and Hinich (1982) depend for their results on a quadratic utility function and a normally distributed random variable representing the nonpolicy differences between the candidates. Coughlin (1986a, 1986b) and Lindbeck and Weibull (1987) assume that each voter cares only about his own income or consumption, making his utility function one dimensional. Consequently, questions arise about the generality of the results associated with the probabilistic theory of elections.

In this paper, we have not tried to argue either for the general existence or nonexistence of equilibrium in probabilistic election theory, but, instead, have tried to look at the question more generally. We have shown that the existence of equilibrium depends on the magnitude of the variance of the random element that represents factors which are probabilistically modelled, the size of the feasible set of candidate policy locations, the salience of policies among voters, the dimensionality of the policy space, and the degree of concavity in voter utility functions.

In deterministic spatial theory, sufficient conditions for equilibrium in two-candidate plurality elections are extremely fragile. Small violations of the conditions destroy equilibrium. By contrast, sufficient conditions for equilibrium in the theory we have discussed exhibit a type of continuity. Several factors are tied to electoral stability, but the existence of stability is a continuous function of their fulfillment.

5. Conclusions

The purpose of this paper is to construct a more general probabilistic theory of elections to show what causes equilibrium and disequilibrium in two-candidate elections. Viewing the candidates as statisticians who can only imperfectly measure the factors that influence voting behavior, we see that reasonable conditions exist for equilibrium and convergence in two-candidate expected vote maximizing contests. We also see what causes these sufficient conditions to break down. The informational conditions that the candidates must satisfy are at least as reasonable as those for deterministic spatial theory.

Prospective uncertainty among voters, reduced policy salience, risk averse voters, and restrictions on the size of the feasible set of policy locations or the dimensionality of the policy space are all stabilizing factors in two-candidate elections. The degree to which voters are 'all over the map' about such factors as the nonpolicy attributes of candidates is important in bringing stability to the policy positions of the candidates. Instead of bemoaning the inability of voters to agree about differences in candidate characteristics, perhaps we should be glad for such disagreements.

The shifting mood of the crowd is a dominant theme in Shakespeare's *Coriolanus*. Coriolanus, standing for Consul of Rome, recognizes that it is pointless to change his stands in response to shifts in popular opinion and so adopts an attitude of 'noble carelessness' toward the voters. The crowd later turns against Coriolanus, but not until after he wins the election.

Notes

1. Closed and bounded, e.g., an interval [a,b] on the real line.
2. We may also define u as the logarithm of policy utility.

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Appendix

As defined in the text, u_j is the first partial derivative of u with respect to r_j and u_{jk} is the second cross partial derivative of u with respect to r_j and r_k. Also u_r is the m-dimensional column vector of first partial derivatives of u, $u_r = (u_1, \ldots, u_m)$, and the $m \times m$ matrix, denoted H_u, of the u_{jk} elements is the Hessian of u with respect to r.

We now define additional notation. If $F = F(d)$, where $d = u(r) - u(t)$, define $F_r(d)$ as the m-dimensional column vector of first partial derivatives of F(d) with respect to $r = (r_1, \ldots, r_m)$, $F_r(d) = f(d)u_r$. $F_j(d)$ is the first partial derivative of F(d) with respect to r_j and $F_{jk}(d)$ is the second cross partial derivative of F(d) with respect to r_j and r_k. The $m \times m$ matrix, denoted H_F, of the $F_{jk}(d)$ elements is the Hessian of F with respect to r. Besides being twice differentiable, both F and u are continuous.

Proof that condition 1 *is necessary and sufficient for concavity of* V(r). $V(r) = E[F(d)]$, where $d = u(r) - u(t)$. By direct differentiation, $F_r(d) = f(d)u_r$ is an $m \times 1$ column vector of partial derivatives with respect to r. In addition,

$$F_{jk}(d) = f'(d)u_j u_k + f(d)u_{jk} \tag{A1}$$

It follows, then, that the Hessian of F(d), H_F can be expressed as the $m \times m$ symmetric matrix

$$H_F = f'(d)u_r u_r^T + f(d)H_u \tag{A2}$$

For any nonzero $m \times 1$ column vector x, we can write the quadratic form

$$x^T H_F x = f'(d)[x^T u_r]^2 + f(d)x^T H_u x \tag{A3}$$

From Condition 1, the expectation of the right-hand side of (A3) is nonpositive. It follows that H_V, the Hessian matrix of V(r), is negative semi-definite, and so V(r) is concave in r. Thus, Condition 1 is a sufficient condition for concavity of V(r).

If $E(x^T H_F x) = x^T H_V x \leq 0$ then Condition 1 holds. Given that F and u are continuous and twice differentiable, if V(r) is concave then H_V is negative semi-definite. Thus, Condition 1 is a necessary condition for concavity of V(r). Q.E.D.