On Plott's pairwise symmetry condition for majority rule equilibrium*

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In a well-known paper, Plott (1967) establishes sufficient conditions for equilibrium in multidimensional choice spaces under majority-rule voting. The most important and best known of these conditions is pairwise symmetry, which states that all nonzero utility gradients at the equilibrium must be divisible into pairs that point in opposite directions. If there is an odd number of voters and exactly one voter's ideal point (zero gradient) at the equilibrium or an even number of voters and no ideal point at the equilibrium, pairwise symmetry is also necessary. If the number of voters is even and one or more ideal points are at the equilibrium, or if the number of voters is odd and two or more ideal points are at the equilibrium, then Plott's conditions are sufficient for the existence of majority rule equilibrium *but they are not necessary.*

This last result is commonly misunderstood. Many references to Plott characterize his conditions as either necessary or necessary and sufficient (Simpson, 1969; McKelvey and Ordeshook, 1976; Shepsle, 1979; Cohen 1979; and Denzau, Mackay and Weaver, 1980) with no mention of the additional assumption required for this statement to be correct. Thus, many public choice theorists appear to believe as Shepsle (1979: 28) states, that 'for equilibrium to obtain, preferences must exhibit an extremely precise symmetry....' Or, as McKelvey and Ordeshook (1976: 1172) state, 'unless the assumption of unidimensional or symmetrically distributed preference is satisfied, the solution to the election game posited by spatial theory does not generally exist.' Scholars familiar with McKelvey's (1976, 1979) results concerning global cycling in the absence of equilibrium remain with the impression that in the multidimensional spatial model, only two cases exist under unrestricted majority rule. In the first case, voter utility gradients exhibit pairwise symmetry around a single point, and, consequently, equilibrium exists. In the second case, voter utility gradients do not exhibit pairwise symmetry, no equilibrium exists, and the entire space is engulfed in a majority

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rule cycle. Besides pointing out this confusion, we wish to pursue a matter raised by Slutsky (1979).

Defining N(0) as the number of voters with zero utility gradients at the equilibrium, N(0) = 1 is a minimum bound for an odd number of voters, and N(0) = 0 is a minimum bound for an even number of voters. Given this result, Slutsky (1979: 1123) observes: 'For sufficiently large N(0), no symmetry requirements are imposed while for N(0) at the minimum necessary, extreme symmetry must occur. For N(0) between these bounds, some less extreme

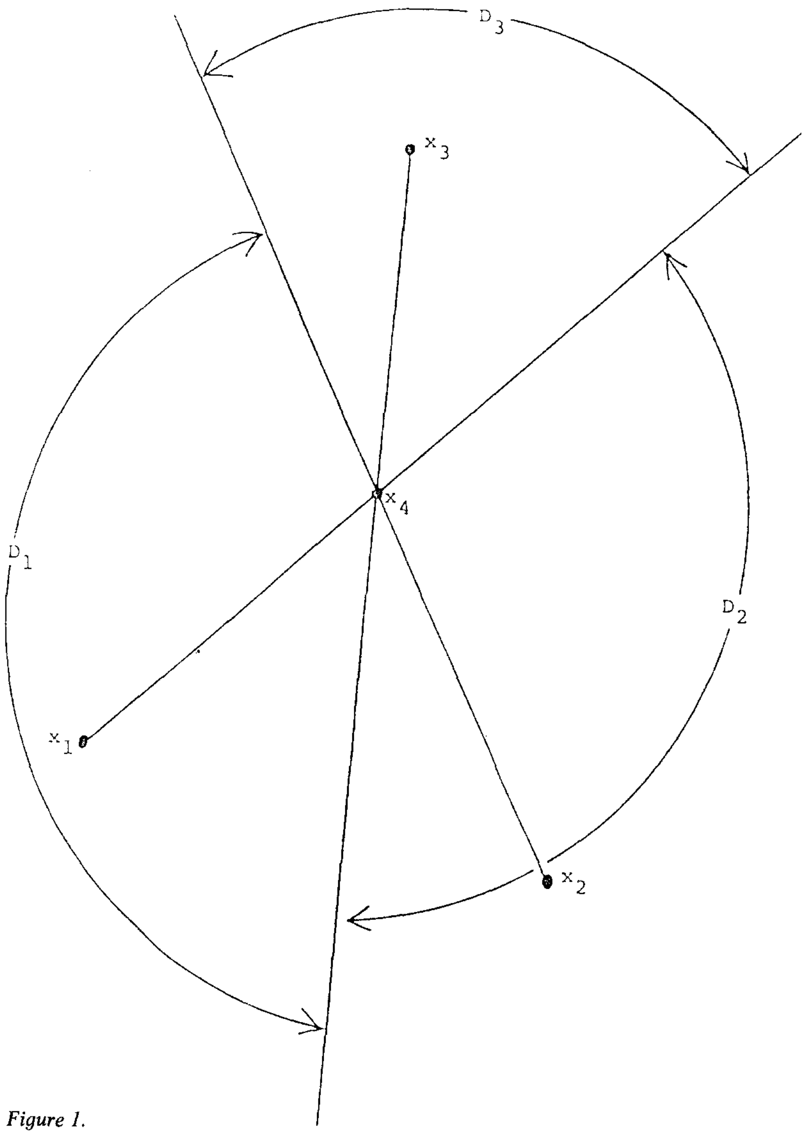

Figure 1.

symmetry on the nonzero gradients is needed.' Here, we construct an example of a majority rule equilibrium with $N(0) = 1$ and an even number of voters, to shed light on the question of how much asymmetry of nonzero gradients can exist when $N(0)$ is one more than the minimum number necessary.

Figure 1 depicts such an example. The choice space is E^2, there are $n = 4$ persons, and each person's preference is based on simple Euclidean distance, representable by circular indifference contours. Figure 1 shows the four ideal points. Equilibrium is defined as in Plott (1967) and in Slutsky (1979), as a point that is at least as good for a majority as any other. This is equivalent to the definition of a dominant point in Davis, DeGroot and Hinich (1972). For $n = 4$, at least three voters constitute a majority. Thus, x_4 is a majority rule equilibrium.

We can use Slutsky's necessary and sufficient condition to check that x_4 is an equilibrium. For any convex pointed cone D^+ with x_4 as the origin and D^- the negative of D^+, the condition is that $|N(D^+) - N(D^-)| \leqslant N(0) = 1$, where $N(D^+)$ is the number of nonzero gradients contained in D^+ and $N(D^-)$ the number in D^-. This method requires checking an infinite number of such cones, as Slutsky points out.

Davis, DeGroot and Hinich (1972) provide an easier way to check for equilibria. They show that a point such as x_4 is an equilibrium (dominant point) if and only if any hyperplane containing x_4 divides the voter ideal points such that at least one-half lie on either closed side of the hyperplane. Clearly, any line passing through x_4 has this property.

The ideal points in Figure 1 plainly do not satisfy Plott's pairwise symmetry condition. Indeed, no two nonzero gradients point in opposite directions. But what can we say about the degree of symmetry required in this case? Slutsky's condition on cones is now useful. Fixing x_4 and any two of the three ideal points not at equilibrium, what restrictions exist on the location of the remaining ideal point if x_4 is to remain in equilibrium? If x_2, x_3, and x_4 are fixed, x_1 can lie anywhere in D_1, the convex cone with origin x_4 bounded by the negative gradients at x_4 of voters 2 and 3. Similarly, if the remaining ideal points are fixed, x_2 can lie anywhere in D_2; and x_3 anywhere in D_3. However, suppose that x_1 lies in the interior of D_2. Then it is possible to construct a smaller convex cone $D_2* \subset D_2$ such that $|N(D_2^+*) - N(D_2^-*)| = 2 > N(0) = 1$.

From a Nash equilibrium standpoint, x_4 is a fairly robust equilibrium, since there is a large area within which each ideal point not at equilibrium can lie without upsetting the equilibrium. Consequently, Shepsle's (1979: 28) concern, voiced by others, that majority rule equilibrium 'is extremely sensitive to slight perturbations' is not entirely justified.

An interesting question remains for $N(0) = 1$ and an even number of voters. How does the necessary degree of nonzero gradient symmetry respond to an increase in the number of voters? Figure 2 for eight voters with circular indifference contours suggests an answer. As the number of voters increases

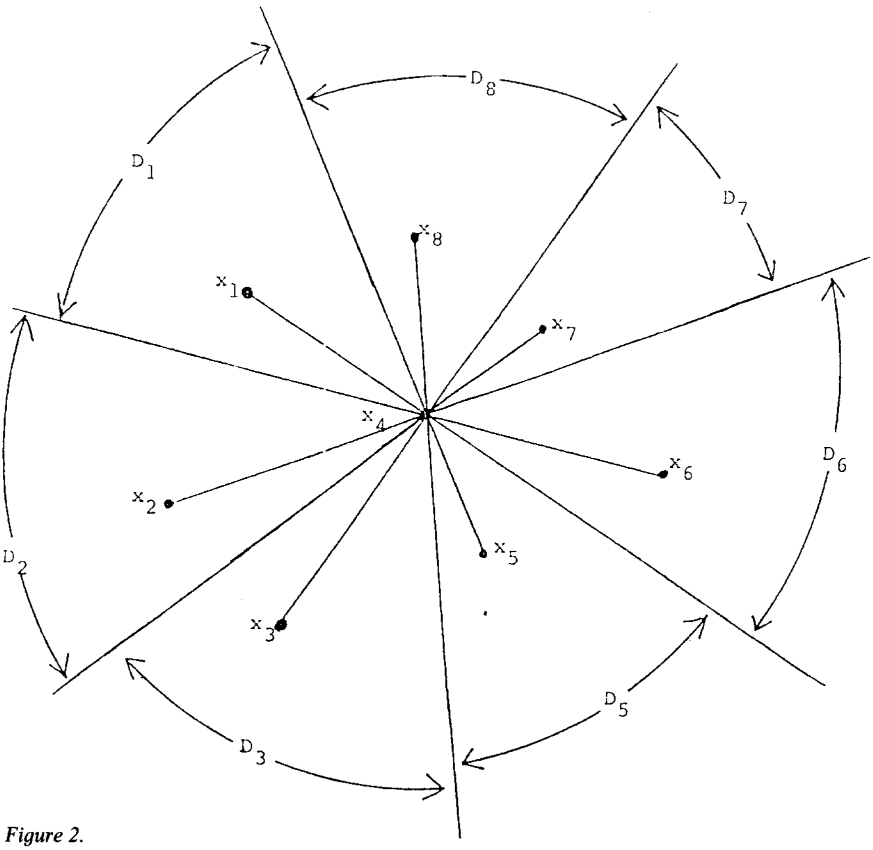

Figure 2.

(for N(0) = 1 and an even number of voters), the necessary degree of nonzero gradient symmetry approaches a variant of Plott's pairwise symmetry condition. For N(0) = 1, every pair of ideal points not at equilibrium requires at least one ideal point contained in the convex cone defined by the negative gradients at the equilibrium of these two ideal points. As the addition of ideal points in the choice space decreases the distance between each adjacent pair of ideal points, the region shrinks within which the necessary ideal point in the negative gradient cone must lie. For each pair of ideal points, in the limit, there must be at least one ideal point in an opposite direction from the equilibrium. For N(0) > 1 the necessary degree of symmetry will increase more slowly as the number of voters increases. If N(0) is at least one-half the number of voters, then no degree of symmetry on the nonzero gradients is required. Thus for fixed N(0), the size of the voting body is directly related to the necessary degree of nonzero-gradient symmetry.

We thus reach three conclusions. *First*, many readers have misread Plott's equilibrium conditions, leading to a widespread misunderstanding that in multidimensional spatial models of unrestricted majority rule voting, pairwise symmetry of nonzero utility gradients at the equilibrium is necessary to avoid global cycling. *Second,* if even one more ideal point than necessary is located at the equilibrium, then no two nonzero gradients have to point in opposite directions. *Finally,* the smaller the number of voters, the less sensitive is equilibrium to perturbations of individual ideal points. The delicate balance among ideal points not at equilibrium that is commonly thought to be necessary to preserve equilibrium is not as delicate as widely believed. This is not to say that equilibrium is a likely occurrence. Rather, the precise relationship between majority-rule equilibrium and the necessary distribution of individual ideal points (or the arrangement of nonzero utility gradients at equilibrium) is not as simple or as neat as many believe it to be.

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