Macroeconomic Dynamics, **7**, 2003, 636–643. Printed in the United States of America.
DOI:10.1017.S1365100503020194

NOTE

RISK WHEN SOME STATES ARE LOW-PROBABILITY EVENTS

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One usually assumes that the joint probability distribution is known or that agents will use Bayesian updating to estimate the true probabilities after a number of trials when the states of nature are finite in classical decision theory under uncertainty. If there are important states that have very low probabilities of occurrence, then each agent must make a subjective assessment of the probability distribution until a sufficient number of outcomes are observed in order to generate a precise estimate of the probability distribution. If one assumes that all agents know the states and their payoffs, the probability distribution is stationary, and they observe all outcomes that unfold over time, then it will take at least 10 times the mean time between occurrences of the lowest probability event in order to generate enough outcomes that all agents share the same objective knowledge of the distribution. The mean time of recurrence depends on both the probability distribution and the time unit used between recordings of the observations. If at least one state has a low probability of occurrence, then the time to convergence will exceed the period of stationarity of the process.

Keywords: Risk, Low-Probability Events

1. INTRODUCTION

The states of nature are finite in the classical decision theory of choice under uncertainty. One either assumes that the joint probability distribution is known or that agents will use Bayesian updating to estimate the true probabilities after a number of trials [La Valle (1970)]. If there are important states that have very low probabilities of occurrence, then each agent must make a subjective estimate of the probability distribution until a sufficient number of outcomes are observed to generate a precise estimate of the true probabilities. Assume that all agents know the states and their payoffs, that they also observe all the outcomes that unfold over time, and that the joint distribution is stationary over time.

I show that it takes at least 10 times the mean time between realizations of the lowest probability event in order to generate enough outcomes that all agents share the same objective knowledge of the distribution. The mean time of recurrence of

I thank the two referees for their constructive criticisms and helpful comments. Professor Mordecai Kurz helped me recognize the importance of stationarity in my argument. Professor John Dalle Molle carefully read the manuscript. I thank him for his help and for improving my English. Address correspondence to: Melvin J. Hinich, 556 Burdine Hall, Mail Code A1800, University of Texas at Austin, Austin, TX 78712, USA; e-mail: hinich@mail.la.utexas.edu.

 1365-1005/03 $12.00

a state depends on both the probability distribution and the time unit between observations.

The relationship between the mean time to get a sample of rare states and the precision of estimation of state probabilities has been overlooked in the application of Bayesian learning in social science theory building. See, for example, Feldman (1987), Jordan (1991), Kalai and Lehrer (1993), and Kim and Yannelis (1997).

I show that subjectivity persists when agents must employ Bayesian updating of subjective probabilities in decision problems when there are low-probability states. The importance of the time unit in the convergence to a precise estimation of rare events is important for modeling financial markets.

The next section presents a simple economy made up of risky assets whose payoffs are determined by a probability distribution for a finite number of states of nature. This simplistic model is used to show how subjectivity persists when at least one state has a low probability of occurrence.

2. SIMPLE MODEL OF RISKY ASSETS

Suppose that an economy has $j = 1, \ldots, J$ risky assets. Asset j can be in one of K_j states at the end of each discrete time point t, which for purposes of exposition will on the hour. Each asset will have a payoff on the hour, 24 hours each day of the year, for a total of 8,760 payoffs per year. These payoffs can be negative as well as positive. All agents observe all outcomes in the economy starting at an initial time t_0 when the market starts functioning. There are no *realizations* of the states before time t_0.

Picture this economy as that of a simple agricultural society with fixed land and static technology. The land is divided into plots that are owned by individuals. The society's law allows and supports tradable property rights. The source of risk for each agricultural asset is the vagaries of the weather, crop disease, and parasites. The assumption of a stationary technology and fixed land removes the growth dynamics from this simple model.

The nonagricultural industrial and service economy is also static and will be ignored. The only service to be factored later in this work is a banking sector that makes fixed-term loans to the landowners.

Each asset has a payoff distribution that depends on the state of nature that exists on the hour. If asset j is in state k_j, then it will pay the owner of that asset x_{jk} with probability π_{jk} for any time t. The x_{jk} are the realizations of this stationary discrete-time multivariate stochastic process whose joint probability distribution is $\pi_j = (\pi_{j1}, \ldots, \pi_{jK_j})$. To simplify calculations, assume that the payoff outcomes are independently distributed realizations over time, but not across assets.

All agents in the economy know the states and their payoffs and they observe all the outcomes that occur after the payoffs begin at time t_0. This is the common knowledge set in this idealized model world.

A probability distribution is not observable. It can be estimated from a sequence of independently distributed observations of the outcomes of the stochastic process.

Suppose that each agent uses a subjective prior probability distribution for the true distribution π_j in order to make decisions about holding assets. There is *no* restriction on agent communication in this world but the classical theory of decisionmaking under risk does not deal with the psychology and sociology of probability assessment. Probabilities in the theory used in this work are treated as individual preference orders are used in economics.

Savage (1954) used the term *personal probability* for the probability values used by a person making decisions. An individual's personal probability is updated by Bayes' rule as observations are made from the conditional probability of the payoff conditional on the state. Savage makes a clear comparison between his theory of personal probability based decisionmaking and the standard frequentist concept of probability theory. The term *personal probability* has lost out to the now standard term *a priori* or *prior probability*. The word *personal* better reflects the distinction between the standard objective probability approach and Savage's approach to the foundations of statistics. The subjective probabilities are personal.

The concept of personal probabilities was indirectly challenged by Aumann (1976) in an influential short paper where he proves that "if two people have the same priors, and their posteriors for an event A are common knowledge, then these posteriors must be equal." He then argues for the assumption of equal priors for different people by distorting the arguments of Harsanyi (1967–1968) about information and inventing the "Harsanyi doctrine." It really does not matter what Harsanyi argues about information since both papers are about theory and not about scientific evidence about how people use and share information in real decisionmaking. The agents in the model in this paper are making personal decisions, not collective ones.

Recall the furor about what was to happen in early 2000 as a result of the Y2K calendar-field problem for computers. A number of respected financial analysts and computer experts took very different public positions on the Y2K impact on the economy. One can always argue that they were not revealing their true beliefs. However, if one rereads the financial journals for the latter part of 1999, then it is easy to see that there was considerable stated disagreement about the likelihood of an economic disaster. A simple argument for what happened is that many experts had different personal probabilities upon which they acted. An excellent review of the literature on the diversity of probability beliefs is presented by Kurz (1997).

3. BAYESIAN UPDATING OF SMALL PROBABILITIES

The asset indexes the states and probabilities in order to show that this seemingly simple discrete state world has a large number of parameters if there are many assets, even if there are only two states for each asset. Since the point is made, the states are now assumed to hold for each asset, and so, the asset index on the state is now unnecessary. The outcomes depend on the asset as well as the state so that the outcome is still doubly indexed.

Suppose that there is a state k_0 whose probability $\pi(k_0)$ is very small. If the probability $\pi(k_0) = 10^{-5}$, for example, then this event will occur, on average, every 11.42 years based on the assumption of 8,760 payoffs per year and a stationary probability distribution. Assuming that our agents can function as classical as well as Bayesian statisticians, if nine occurrences of state k_0 occur in a 100-year period, the sample proportion estimate of $\pi(k_0)$ is $\hat{\pi}(k_0) = 9/876{,}000 = 1.03 \times 10^{-5}$. The 95% approximate confidence interval is $(0.36 \times 10^{-5}, 1.70 \times 10^{-5})$. This interval contains the true value of $\pi(k_0)$ but with a 67% plus- or minus spread. It will take about 100 years for agents to observe sufficient outcomes of this low-probability state to estimate the probability to an accuracy of an order of magnitude. This is truly a *long time*!

The stationarity of the joint distribution is crucial for this convergence result to hold. If the distribution shifts either abruptly or gradually over a period of 10 to 20 years, then the agents will always have heterogeneous subjective probability estimates. If $\pi(k_0)$ shifts, say, from 10^{-5} to $5 \cdot 10^{-6}$, then the agents will not detect this shift before another occurs. Time plays an important role in this simple statistical problem when there is at least one low-probability state.

This convergence result is consistent with the theory of rational belief developed by Kurz (1997) and Kurz and Motolese (2001). In the theory of rational belief, agents have different beliefs about the probabilities of outcome from an endogenous nonstationary economic system. The existence of a nonstationary environment is central to the rational-belief theory and thus the speed of convergence matters. Indeed, slow learning is the foundation of this theory.

The confidence-interval calculations present an insight into the rarity of this event but they do not directly address the Bayesian updating issue. To work this problem into a Bayesian update problem, suppose that state k_0 does not occur in a 100-year period. The conditional probability of this event $E = \{k_0 \text{ does not occur in 100 years}\}$ is $[1 - \pi(k_0)]^n$, where n is 876,000.

Suppose that agent *alpha* uses the prior density function $p_\alpha[\pi(k_0)] = \alpha[1 - \pi(k_0)]^{\alpha-1}$, where $\alpha > 1$. This prior density makes sense when agent *alpha* knows that the true value of $\pi(k_0)$ is small. Consequently, the agent has the posterior probability density

$$p_\alpha[\pi(k_0)|E] = (\alpha + n)[1 - \pi(k_0)]^{\alpha+n-1}. \tag{1}$$

The mean of this posterior probability density is $(\alpha + n + 1)^{-1}$. The parameter α serves as a subjective augmentation of the sample size n. This example shows that the prior subjective probability distribution can strongly influence the posterior distribution for large n when $\pi(k_0)$ is very small.

The independence assumption is made for simplicity. If there is positive dependence over time, then the number of observations required to reach an accurate estimate is longer than what is presented above.

The posterior density of $\pi(k_0)$ when the first outcome of state k_0 occurs will put more mass around the true probability than does (1). As more outcomes are observed, the posterior probability density will concentrate around the true probability. This simple argument shows that the time span for updating to a precise posterior will take a long time. These results will be the same if $\pi(k_0) = 10^{-7}$ and the time interval between observations is 0.6 minute. The objective reality for small-probability outcomes requires a long time of observation when the outcome timescale is not small.

Agents will trade assets using subjective probability assessments. Even if only one state of nature has a low probability, the subjectivity leaks into the expected value since the probabilities of all the states must sum to one. It is now time to address the issue of "so what." The next section addresses some of the implications for market decisionmaking when there are heterogeneous beliefs about the $\pi(k)$ probabilities. The simplicity of the multinomial model makes it easy to see the basic relationships between prior probability beliefs, the time between observations, and the length of time it takes for agents to converge to a common posterior probability distribution.

4. MARKETS AND LOTTERIES

Let $\pi_\alpha(k)$ denote agent *alpha*'s subjective probability that state k occurs at time t. There is a subjective component for each state k since $\pi_\alpha(k_0)$ is subjective and $\sum_{k=1}^{K} \pi_\alpha(k) = 1$. Assuming that at least one state has a sufficiently low probability such that the mean time between outcomes is at least 10 years, the subjective probability updating takes so long that a time index is not needed for the agent's subjective probability distribution. In other words, the $\pi_\alpha(k)$'s can be treated as stationary over a period of about 100 years, which is longer than the stationarity of the whole economy. Over time, some assets disappear, others change their character, and new assets come into existence. The mean time between rare events is longer than what stationarity exists in the market process.

All agents know that the other agents use subjective probabilities in their decision process. They know that there is no objective method to accurately estimate the low probabilities. Each individual makes idiosyncratic estimates based on common knowledge. Assume that no two agents have identical subjective probability distributions.

Let p_{jt} denote the market price of asset j at time t. Assume for now that each agent has a linear cardinal utility function for wealth. The assumption of risk-neutral utility function is made to allow simple calculations of examples. Standard concave utility-based portfolio theory is invoked in the next section to address agent borrowing to acquire assets. A generalization to standard risk avoidance makes the portfolio calculations a bit more complicated without altering the validity of the argument that subjectivity plays a crucial role in the market process.

Agent *alpha*'s expected utility for one unit of asset j at time t is

$$\mu_{\alpha j} = p_{jt} \sum_{k=1}^{K} \pi_\alpha(k) x_{jk}. \tag{2}$$

Agent *alpha* prefers the asset with the highest expected value. Variance does not affect the preference since the agent is risk neutral. If there are several assets with the same expected value, the agent is indifferent among them. Assume for simplicity that there is only one asset at the maximum.

An example is helpful here. Suppose that there are only four states and two assets in the economy. Suppose that the outcomes of asset *1* and asset *2*, respectively, are

$$\begin{aligned} x_{11} &= -10^5, \quad x_{12} = 0, \quad x_{13} = 10, \quad x_{14} = 10^4; \\ x_{21} &= -10^6, \quad x_{22} = 0, \quad x_{23} = 10^2, \quad x_{24} = 10^5. \end{aligned} \qquad (\mathbf{3})$$

Suppose that agent *alpha*'s subjective probability distribution is

$$\pi_\alpha(1) = 10^{-5}, \;\; \pi_\alpha(2) = 0.3 - \pi_\alpha(1), \;\; \pi_\alpha(3) = 0.7 - \pi_\alpha(4), \;\; \pi_\alpha(4) = 10^{-4}. \qquad (\mathbf{4})$$

Then, agent *alpha*'s expected utilities for assets *1* and *2*, respectively, at time t are $\mu_{\alpha 1} = 6.999 p_{1t}$ and $\mu_{\alpha 2} = 60.99 p_{2t}$.

Suppose agent *beta*'s subjective probability distribution is

$$\pi_\beta(1) = 10^{-4}, \;\; \pi_\beta(2) = 0.3 - \pi_\beta(1), \;\; \pi_\beta(3) = 0.7 - \pi_\beta(4), \;\; \pi_\beta(4) = 10^{-4}. \qquad (\mathbf{5})$$

Then, agent *beta*'s expected utilities for assets *1* and *2*, respectively, at time t are $\mu_{\beta 1} = -2.001 p_{1t}$ and $\mu_{\beta 2} = -29.01 p_{2t}$.

If $p_{1t} < p_{2t}$, then agent *alpha* prefers asset *2* to asset *1* whereas agent *beta* prefers asset *1*. Both agents possess the same information about the states and outcomes and both use Bayesian updating to modify their prior probabilities. Yet they have different expected values for the two assets. Note that *alpha* has positive expected values whereas *beta* has negative expected values for the two assets. If *beta* owns either asset, he will want to trade it for an asset with positive expected value.

Assume that all agents have strictly concave utility functions rather than linear utility functions. Standard portfolio theory now applies for each period [portfolio theory when agents have quadratic utility functions is developed by Markowitz (1959)].

Assume that an agent has a budget per period for investment. The agent allocates this fixed amount per period to buying assets along with proceeds from sales of assets per period. This assumption simplifies the decisionmaking problem by separating the consumption versus savings multiperiod problem [Lippman & McCall (1981, Sect. 4)]. Each agent during every period selects a portfolio of assets that maximizes his expected utility, given his budget constraint.

This decision process continues each period. Since the subjective probabilities are slowly changing, individual portfolios change from period to period due to changes in prices and budgets. The subjective probabilities of the agents are an important component of the market transaction process.

Assume that there are sufficient agents in the economy operating in an efficient market-clearing process so that an equilibrium market price of each asset is achieved at each period. What is different in this theory from the standard theory is that agents with the same utility function, the same budget, and the same information will have different expected values of an asset.

Recall that all agents observe all states and know the states and their payoffs. They start with different personal beliefs about the low-probability states. Trading in this world is different from that discussed by Milgrom and Stokey (1982). The differences in personal probabilities are *not* due to differences in information about the true probability distribution. All agents are uninformed about the true probability of the unlikely state. Each individual acts as if he has a personal probability distribution and update using Bayes' rule. Since all are uninformed about the true probabilities, they cannot convince one another about the truth since all know that they all do not know the truth.

5. RESERVE REQUIREMENTS UNDER UNCERTAINTY

Assume that agents can borrow to buy more assets by using some of their assets as collateral. If the lenders are regulated banks, then the subjective element of the market poses a problem for bank regulation.

Suppose that an agent wants to borrow to buy an asset whose expected value is greater than the interest rate for the period. The agent borrows the price of the asset using as collateral some low-risk assets in his portfolio. The term "low risk" is used rather than the standard term "risk-free" because no assets are truly risk-free. U.S. government bonds are low risk at present but there is some very low-probability that the U.S. government will suddenly adopt some policy that results in default of the bonds. The main thesis of this paper is that low-probability states play a subtle role in the market process.

Bankers share the subjectivity inherent in the economy. Individuals have different subjective probabilities over the states. A loan officer has a different expected value from the borrower for the asset to be purchased and the collateral assets. The owners of the bank shares and the bank regulators have different evaluations.

Suppose that a number of large banks underestimate the probability of a "bad" state by an order of magnitude or two. They will loan money to agents with the same or lower subjective probability of that state provided that the expected value is in their favor given the interest they can charge the agent. Each agent may borrow a small amount but if there are a large number of optimistic agents, then the bank can be in a more exposed position, given the true probability distribution as compared with their subjective assessment.

If the "bad" state occurs, then the banks may be in trouble. If bank problems have no impact on the general economy but only on the owners of the bank, then the "bad" outcomes are the downs of doing business. However, if the bad loans weigh on the economy due to the fractional reserve nature of banking, then the issue of proper reserve requirements must reflect the incomplete knowledge of the true probability distribution.

Individual choice market mechanisms do not provide enough information to solve this problem. There is no simple solution to the problem of setting reserve requirements when the probability distribution of the states of nature is not exactly known.

The fundamental uncertainty inherent in the Bayesian updating of prior probabilities for finite observations over a finite time results in a collectivization of this banking risk. The specific collective decision rule used depends on the politics of the society.

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