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A Class Test for Fractional Integration

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Abstract

Diebold and Rudebusch (1991) and Haubrich (1993) argue that, when income follows a fractionally differenced process, the Deaton's excessive smoothness paradox can be resolved. A key to the success of their result relies on a valid test for fractional integration. However, most of the tests in the literature are nested within fractional alternatives. This paper designs a new test for a more general hypothesis that the true data generating process is indeed fractionally integrated. The test is applied to the real disposable income per capita of the U.S. and the real quarterly GDP data of the G7 industrial countries.

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1. Introduction

Introduced by Granger and Joyeux (1980), the fractionally integrated model has found wide applications in various disciplines. The model has been applied to asset pricing models (Ding et al., 1993), stock returns (Lo, 1991), interest rates (Shea, 1991; Backus and Zin, 1993; Crato and Rothman, 1994) and inflation rates (Hassler and Wolters, 1995). A fractionally integrated process is mainly characterized by the differencing parameter which governs the memory property of the process. A positive value of the differencing parameter implies that the process has long memory. It has long been recognized that many macro-economic time series display long memory property.

There has been a great stride forward in the estimation of the long memory model in the past two decades (Geweke and Porter-Hudak, 1983; Li and McLeod, 1986; Sowell, 1992; Hurvich and Ray, 1995; Chong, 2006; Mayoral, 2006). Tests for long memory have also been examined (Cheung, 1993; Wright, 1999; Chen and Deo, 2004). For a comprehensive review of the literature in long memory and fractional integration, one is referred to Baillie (1996), Henry and Zaffaroni (2002) and Robinson (2003). Despite the extensive applications of the process, the development of a test on whether the observations are generated by a fractionally integrated process is heretofore in its infancy stage. In light of this, this paper proposes a new test which can distinguish fractionally integrated processes from other time series processes. We also derive the asymptotic distribution of the test and simulate its finite-sample counterpart. The test is applied to the U.S. per capita real disposable income and the quarterly real GDP data of the G7 industrial countries.

The remainder of this paper is organized as follows: Section 2 presents the model. Section 3 suggests a new test for fractional integration and derives its asymptotic properties. Section 4 examines the performance of the test in finite samples. Section 5 provides empirical applications of the test and Section 6 concludes the paper.

2. The Model

A time series process $\{y_t\}_{t=1}^{T}$ is said to be generated from an $ARFIMA\,(p,d,q)$ process if

$$\phi\left(L\right)\left(1-L\right)^{d}y_t=\theta\left(L\right)u_t,\qquad t=1,2,...,T, \tag{2.1}$$

where $\{u_t\} \sim i.i.d.\,(0,\sigma^2)$, L is the lag operator, $\phi(L) = 1 - \phi_1 L - ... - \phi_p L^p$, $\theta(L) = 1 - \theta_1 L - ... - \theta_q L^q$. If d is not an integer, then the process is said to be fractionally integrated.

The fractional difference operator $(1-L)^d$ is defined by its Maclaurin series

$$(1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(-d)\,\Gamma(j+1)} L^j, \tag{2.2}$$

where $\Gamma(x)$ is the Euler gamma function defined as

$$\Gamma(x) = \int_0^{\infty} z^{x-1} \exp(-z)\, dz \qquad \text{for } x > 0,$$

$$\Gamma(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(x+k)\,k!} + \int_1^{\infty} z^{x-1} \exp(-z)\, dz \qquad \text{for } x < 0, x \neq -1, -2, -3, ...$$

The process is stationary if $d < 0.5$. It can be represented by a $MA(\infty)$ process defined as

$$y_t = \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(d)\,\Gamma(j+1)} u_{t-j}, \qquad (t = 1, 2, ..., T). \tag{2.3}$$

For simplicity, this paper discusses a pure $I(d)$ process, i.e., an $ARFIMA(0,d,0)$ process. It is well established (Hosking, 1996) that for an $I(d)$ process with $-0.5 < d < 0.5$,

$$\rho_j = \prod_{i=1}^{j} \frac{d+i-1}{i-d} \qquad (j = 1, 2, ...). \tag{2.4}$$

and

$$T^{0.5-d}\overline{y} \xrightarrow{d} N\left(0, \frac{\sigma^2 \Gamma(1-2d)}{(1+2d)\,\Gamma(1+d)\,\Gamma(1-d)}\right), \tag{2.5}$$

where ρ_j is the j^{th} autocorrelation and $\overline{y}$ is the sample mean of $\{y_t\}_{t=1}^{T}$.

Given the value of the differencing parameter, the standardized spectral density is equal to

$$f(\lambda|d) = \frac{1}{2\pi}|1-\exp(-i\lambda)|^{-2d} \qquad -\pi \leq \lambda \leq \pi$$
$$= \frac{1}{2\pi}\left|2\sin\left(\frac{\lambda}{2}\right)\right|^{-2d}, \tag{2.6}$$

and the standardized spectral distribution is

$$F(\lambda|d) = 2\int_0^\lambda \frac{1}{2\pi}\left(2\sin\left(\frac{\upsilon}{2}\right)\right)^{-2d} d\upsilon, \qquad 0 \leq \lambda \leq \pi. \tag{2.7}$$

Suppose we run a regression of y_t on $y_{t-1}, y_{t-2}, .., y_{t-n}$. Let

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ y_{T-1} \\ y_T \end{pmatrix}, \qquad X_n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ y_1 & 0 & \cdots & 0 \\ y_2 & y_1 & \cdots & 0 \\ \vdots & y_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & y_1 \\ \vdots & \vdots & \cdots & \vdots \\ y_{T-1} & y_{T-2} & \cdots & y_{T-n} \end{pmatrix},$$

$$\widehat{\boldsymbol{\beta}}(n) = \begin{pmatrix} \widehat{\beta}_{n,1} & \widehat{\beta}_{n,2} & \cdots & \widehat{\beta}_{n,n-1} & \widehat{\beta}_{n,n} \end{pmatrix}' = (X_n' X_n)^{-1} X_n' Y.$$

Dividing each element in $X_n' X_n$ and $X_n' Y$ by $\sum_{t=2}^{T} y_{t-1}^2$ and take probability limit, we have

$$\widehat{\boldsymbol{\beta}}(n) \xrightarrow{p} \Phi(n-1)^{-1}\boldsymbol{\rho}(n) = \boldsymbol{\beta}(n),$$

where

$$\Phi(n-1) = \begin{pmatrix} 1 & \rho_1 & \cdots & \rho_{n-1} \\ \rho_1 & 1 & \cdots & \rho_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n-1} & \rho_{n-2} & \cdots & 1 \end{pmatrix} \tag{2.8}$$

is an $n \times n$ Toeplitz matrix,

$$\boldsymbol{\rho}(n) = \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_{n-1} & \rho_n \end{pmatrix}', \tag{2.9}$$

$$\boldsymbol{\beta}(n) = \begin{pmatrix} \beta_{n,1} & \beta_{n,2} & \cdots & \beta_{n,n-1} & \beta_{n,n} \end{pmatrix}'. \tag{2.10}$$

An $I(d)$ process for $-0.5 < d < 0.5$ has a feature that, if it is approximated by an $AR(n)$ model via a regression, then the probability limits of the AR's coefficient estimates are functions of d and n. Specifically, as $T \to \infty$,

$$\widehat{\boldsymbol{\beta}}(2) \xrightarrow{p} \Phi(1)^{-1}\boldsymbol{\rho}(2) = \begin{pmatrix} \rho_1 \dfrac{1-\rho_2}{1-\rho_1^2} & \dfrac{\rho_1^2-\rho_2}{-1+\rho_1^2} \end{pmatrix}' = \begin{pmatrix} \dfrac{2d}{2-d} & \dfrac{d}{2-d} \end{pmatrix}',$$

$$\widehat{\boldsymbol{\beta}}(3) \xrightarrow{p} \Phi(2)^{-1}\boldsymbol{\rho}(3) = \begin{pmatrix} \dfrac{3d}{3-d} & \dfrac{3d(1-d)}{(3-d)(2-d)} & \dfrac{d}{3-d} \end{pmatrix}',$$

$$\widehat{\boldsymbol{\beta}}(4) \xrightarrow{p} \Phi(3)^{-1}\boldsymbol{\rho}(4) = \begin{pmatrix} \dfrac{4d}{4-d} & \dfrac{6d(1-d)}{(4-d)(3-d)} & \dfrac{4d(1-d)}{(4-d)(3-d)} & \dfrac{d}{4-d} \end{pmatrix}'.$$

In general, we have

$$\widehat{\beta}_{n,j} \xrightarrow{p} -\binom{n}{j}\frac{\Gamma(j-d)\,\Gamma(n-d-j+1)}{\Gamma(-d)\,\Gamma(n-d+1)}. \tag{2.11}$$

3. The Test

Note that the estimated coefficients of y_{t-1} and y_{t-n} converge in probability to $\dfrac{nd}{n-d}$ and $\dfrac{d}{n-d}$ respectively. Thus, if the true process is $I(d)$ and if the sample size is large enough, the first estimate will be about n times the last one. As a result, a test of whether the process follows an $I(d)$ can be constructed based on the elegant relationship between $\widehat{\beta}_{n,1}$ and $\widehat{\beta}_{n,n}$ that

$$\widehat{\beta}_{n,1} - n\widehat{\beta}_{n,n} \xrightarrow{p} 0. \tag{3.1}$$

Towards this end, we can run $(n-1)$ autoregressions $AR(2), AR(3), ..., AR(n)$, and define

$$W(d,n) = (B(n,1) - B(n,n)\Lambda(n))\,\Omega(d)^{-1}\,(B(n,1) - B(n,n)\Lambda(n))', \quad (3.2)$$

where

$$B(n,1) = \left(\begin{array}{cccc} \widehat{\beta}_{2,1} & \widehat{\beta}_{3,1} & \cdots & \widehat{\beta}_{n,1} \end{array} \right),$$

$$B(n,n) = \left(\begin{array}{cccc} \widehat{\beta}_{2,2} & \widehat{\beta}_{3,3} & \cdots & \widehat{\beta}_{n,n} \end{array} \right),$$

$$\Lambda(n) = diag\left(\begin{array}{cccc} 2 & 3 & \cdots & n \end{array} \right),$$

$$\Omega(d) = E\left[(B(n,1) - B(n,n)\Lambda(n))'(B(n,1) - B(n,n)\Lambda(n))\right].$$

The elements of the matrix $\Omega(d)$ depend on $\beta's$, which in turn depend on the value of d. We test

$$H_0 : y_t \sim I(d)$$

against

$$H_1 : y_t \text{ does not follow } I(d)$$

for $-0.5 < d < 0.25$[1].

If the null hypothesis is correct, then there exists a differencing parameter d such that $W(d,n)$ is $O_p(1)$. Otherwise, the test will diverge. To construct the matrix $\Omega(d)$, note that for $l, m = 2, 3, ..., n$, the $(l-1, m-1)^{th}$ element of $\Omega(d)$ can be written as

$$\begin{aligned} & \Omega(d)_{l-1,m-1} \\ = \; & Cov\left(\widehat{\beta}_{l,1}, \widehat{\beta}_{m,1}\right) - Cov\left(\widehat{\beta}_{l,l}, \widehat{\beta}_{m,1}\right) l - Cov\left(\widehat{\beta}_{l,1}, \widehat{\beta}_{m,m}\right) m + Cov\left(\widehat{\beta}_{l,l}, \widehat{\beta}_{m,m}\right) lm, \end{aligned}$$

[1] As far as the estimation is concerned, we allow $-0.5 < d < 0.5$. However, for $d > 0.25$, the distribution of $W(d,n)$ will no longer be Chi-squared but something related to the Rosenblatt distribution as found in Hosking (1996). For simplicity, we assume that $-0.5 < d < 0.25$. Tieslau et al. (1996) and Chong (2000) also assume $-0.5 < d < 0.25$ in their studies.

where

$$Cov\left(\widehat{\beta}_{l,1},\widehat{\beta}_{m,1}\right)=L_1\left(l\right)E\left(\widehat{\boldsymbol{\beta}}\left(l\right)-\boldsymbol{\beta}\left(l\right)\right)\left(\widehat{\boldsymbol{\beta}}\left(m\right)-\boldsymbol{\beta}\left(m\right)\right)'L_1\left(m\right)',$$

$$Cov\left(\widehat{\beta}_{l,l},\widehat{\beta}_{m,1}\right)=L_2\left(l\right)E\left(\widehat{\boldsymbol{\beta}}\left(l\right)-\boldsymbol{\beta}\left(l\right)\right)\left(\widehat{\boldsymbol{\beta}}\left(m\right)-\boldsymbol{\beta}\left(m\right)\right)'L_1\left(m\right)',$$

$$Cov\left(\widehat{\beta}_{l,1},\widehat{\beta}_{m,m}\right)=L_1\left(l\right)E\left(\widehat{\boldsymbol{\beta}}\left(l\right)-\boldsymbol{\beta}\left(l\right)\right)\left(\widehat{\boldsymbol{\beta}}\left(m\right)-\boldsymbol{\beta}\left(m\right)\right)'L_2\left(m\right)',$$

$$Cov\left(\widehat{\beta}_{l,l},\widehat{\beta}_{m,m}\right)=L_2\left(l\right)E\left(\widehat{\boldsymbol{\beta}}\left(l\right)-\boldsymbol{\beta}\left(l\right)\right)\left(\widehat{\boldsymbol{\beta}}\left(m\right)-\boldsymbol{\beta}\left(m\right)\right)'L_2\left(m\right)',$$

$$L_1\left(i\right)=\underbrace{\left(1\ 0\ ...\ 0\ 0\right)}_{i\text{ terms}},$$

$$L_2\left(i\right)=\underbrace{\left(0\ 0\ ...\ 0\ 1\right)}_{i\text{ terms}}.$$

To evaluate $E\left(\widehat{\boldsymbol{\beta}}\left(l\right)-\boldsymbol{\beta}\left(l\right)\right)\left(\widehat{\boldsymbol{\beta}}\left(m\right)-\boldsymbol{\beta}\left(m\right)\right)'$, note that since

$$\begin{aligned}\widehat{\boldsymbol{\rho}}\left(n\right)-\boldsymbol{\rho}\left(n\right) &= \widehat{\Phi}\left(n-1\right)\widehat{\boldsymbol{\beta}}\left(n\right)-\Phi\left(n-1\right)\boldsymbol{\beta}\left(n\right)\\ &= \Phi\left(n-1\right)\left(\widehat{\boldsymbol{\beta}}\left(n\right)-\boldsymbol{\beta}\left(n\right)\right)+\left(\widehat{\Phi}\left(n-1\right)-\Phi\left(n-1\right)\right)\boldsymbol{\beta}\left(n\right)\\ &\quad +\left(\widehat{\Phi}\left(n-1\right)-\Phi\left(n-1\right)\right)\left(\widehat{\boldsymbol{\beta}}\left(n\right)-\boldsymbol{\beta}\left(n\right)\right)\\ &= \Phi\left(n-1\right)\left(\widehat{\boldsymbol{\beta}}\left(n\right)-\boldsymbol{\beta}\left(n\right)\right)+\left(\widehat{\Phi}\left(n-1\right)-\Phi\left(n-1\right)\right)\boldsymbol{\beta}\left(n\right)+O_p\left(T^{-1}\right),\end{aligned}$$

we have

$$\widehat{\boldsymbol{\beta}}\left(n\right)-\boldsymbol{\beta}\left(n\right)=\Phi\left(n-1\right)^{-1}\Delta\left(n\right), \tag{3.3}$$

where

$$\Delta(n) = (\widehat{\boldsymbol{\rho}}(n) - \boldsymbol{\rho}(n)) - \left(\widehat{\Phi}(n-1) - \Phi(n-1)\right)\boldsymbol{\beta}(n) + O_p\left(T^{-1}\right). \qquad (3.4)$$

Hence, $E\left(\widehat{\boldsymbol{\beta}}(l) - \boldsymbol{\beta}(l)\right)\left(\widehat{\boldsymbol{\beta}}(m) - \boldsymbol{\beta}(m)\right)'$ is reduced to

$$\Phi(l-1)^{-1} E(\Delta(l)\,\Delta(m)')\Phi(m-1)^{-1}.$$

To find $E(\Delta(l)\,\Delta(m)'$, note that

$$\begin{aligned}
& \lim_{T\to\infty} TE(\Delta(l)\,\Delta(m)') \\
= \; & C(l,m) - \lim_{T\to\infty} TE\left(\widehat{\Phi}(l-1) - \Phi(l-1)\right)\boldsymbol{\beta}(l)\,(\widehat{\boldsymbol{\rho}}(m) - \boldsymbol{\rho}(m))' \\
& - \lim_{T\to\infty} TE\,(\widehat{\boldsymbol{\rho}}(l) - \boldsymbol{\rho}(l))\,\boldsymbol{\beta}(m)'\,(\widehat{\Phi}(m-1) - \Phi(m-1))' \\
& + \lim_{T\to\infty} TE\left(\widehat{\Phi}(l-1) - \Phi(l-1)\right)\boldsymbol{\beta}(l)\,\boldsymbol{\beta}(m)'\,(\widehat{\Phi}(m-1) - \Phi(m-1)),
\end{aligned}$$

where $C(l,m)$ is an l by m matrix with the $(i,j)^{th}$ element $c_{i,j}$ being given by

$$c_{i,j} = \sum_{s=1}^{\infty}\left(\rho_{s+i} + \rho_{s-i} - 2\rho_s\rho_i\right)\left(\rho_{s+j} + \rho_{s-j} - 2\rho_s\rho_j\right). \qquad (3.5)$$

Thus, the $(i,j)^{th}$ element of $\lim_{T\to\infty} TE(\Delta(l)\,\Delta(m)')$ is given by

$$\begin{aligned}
& \lim\nolimits_{T\to\infty} TE(\Delta(l)\,\Delta(m)')_{i,j} \\
& = c_{i,j} - \textstyle\sum_{h=1,h\neq i}^{l} \beta_{l,h} c_{|i-h|,j} - \sum_{k=1,k\neq j}^{m} \beta_{m,k} c_{|j-k|,i} \\
& + \textstyle\sum_{h=1,h\neq i}^{l} \sum_{k=1,k\neq j}^{m} E\left(\widehat{\Phi}(l-1) - \Phi(l-1)\right)_{i,h} \\
& \times \left(\widehat{\Phi}(m-1) - \Phi(m-1)\right)_{k,j} \beta_{l,h}\beta_{m,k} \\
& = c_{i,j} - \textstyle\sum_{h=1,h\neq i}^{l} \beta_{l,h} c_{|i-h|,j} - \sum_{k=1,k\neq j}^{m} \beta_{m,k} c_{|j-k|,i} \\
& + \textstyle\sum_{h=1,h\neq i}^{l} \sum_{k=1,k\neq j}^{m} c_{|i-h|,|k-j|} \beta_{l,h}\beta_{m,k}.
\end{aligned}$$

Therefore, the elements of the matrix $\Omega(d)$ depend on the $\beta's$ and $c's$, which in turn depend on the value of d. To make the test operational, we have to approximate $\Omega(d)$ by $\Omega\left(\widehat{d}\right)$, where $\widehat{d}$ is a consistent estimator for d. The following theorem states the asymptotic distribution of $W\left(\widehat{d}, n\right)$:

Theorem 1: Given a consistent estimator $\widehat{d}$ for d, the test statistic converges in distribution to a Chi-square distribution with degrees of freedom $(n-1)$ as $T \to \infty$ under the null hypothesis, i.e.,

$$W\left(\widehat{d}, n\right) \xrightarrow{d} \chi^2(n-1). \tag{3.6}$$

Proof. See the appendix.

The remaining problem is to select a consistent $\widehat{d}$. In principle, we can employ any consistent estimator recently proposed in the literature. The estimators can be parametric (Dahlaus, 1989; Sowell, 1992) or semiparametric (Robinson, 1995; Velasco, 1999a, 1999b; Phillips and Shimotsu, 2004). In our case, since we have already obtained $\widehat{\beta}_{j,1}$ and $\widehat{\beta}_{j,j}$, we can utilize this piece of information to estimate d. Note that for $j = 1, 2, 3, ..., n$,

$$\widehat{\beta}_{j,1} \xrightarrow{p} \frac{jd}{j-d},$$

$$\widehat{\beta}_{j,j} \xrightarrow{p} \frac{d}{j-d}.$$

Thus, we have

$$\widehat{d}_{j,1} = \frac{j\widehat{\beta}_{j,1}}{j+\widehat{\beta}_{j,1}}, \tag{3.7}$$

$$\widehat{d}_{j,j} = \frac{j\widehat{\beta}_{j,j}}{1+\widehat{\beta}_{j,j}}. \tag{3.8}$$

In fact, $\widehat{\beta}_{j,j}$ is the estimator for the j^{th} order partial autocorrelation[2] of an $I(d)$ process, and $\widehat{\beta}_{j,1}$ is just j times $\widehat{\beta}_{j,j}$. We suggest a robust and consistent estimator for d by taking the *median* of these estimates. We arrange $\widehat{d}_{j,1}$, $\widehat{d}_{j,j}$, $(j = 1, 2, 3, ..., n)$ in an ascending order. As $\widehat{d}_{j,j} = \widehat{d}_{j,1}$ for $j = 1$, we have a total

[2] For the properties of $\widehat{d}$ based on the partial autocorrelation, one is referred to Chong (2000).

of $(2n-1)$ estimates. For $i = 1, 2, ..., 2n-1$, we denote the i^{th} order statistic as $\widehat{d}_{(i)}$, and define the median estimator of d as

$$\widehat{d} = \widehat{d}_{(n)}. \tag{3.9}$$

Since the mappings in (3.7) and (3.8) are continuous, and all the estimators are consistent, the median estimator is also consistent by the Sandwich Theorem.

4. Monte Carlo Experiments

Experiment 1. This experiment verifies Theorem 1 that $W(d, n)$ is asymptotically Chi-square distributed under the null. Consider the following model:

$$(1-L)^d y_t = u_t, \qquad t = 1, 2, ..., T.$$

$$T = 50, 100, 200, 500$$

$$u_t \sim N(0, 1).$$

$$d = -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2.$$

Tables 1a to 1d report the critical value c of the finite sample distribution of $W(d, n)$ such that

$$\Pr(W(d, n) \leq c) = p,$$

for $T = 50$, 100, 200 and 500 respectively.

For each value of T, d and n, we simulate the test statistic $W(d, n)$ with 100000 replications. The critical values of the Chi-square distribution with degrees of freedom $(n-1)$ are also tabulated for comparison. Observe that the finite sample distribution is justifiably approximated by its limiting distribution. The departure of the critical values of the the test in finite samples from their asymptotic counterparts is small.

Table 1a: Critical values c of $W(d,n)$ such that $\Pr(W(d,n) \leq c) = p$, T=50.

n=2	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(1)$
	.99	6.37	6.44	6.45	6.53	6.50	6.52	6.53	6.63
	.975	4.88	4.93	4.98	4.99	4.97	4.98	4.98	5.02
	.95	3.76	3.79	3.85	3.85	3.81	3.84	3.85	3.84
	.9	2.69	2.71	2.75	2.74	2.70	2.73	2.73	2.71
n=3	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(2)$
	.99	9.29	9.22	9.12	9.02	9.10	9.17	8.99	9.21
	.975	7.50	7.41	7.36	7.34	7.35	7.38	7.23	7.38
	.95	6.11	6.05	6.00	6.04	6.02	6.03	5.96	5.99
	.9	4.71	4.66	4.70	4.69	4.68	4.68	4.65	4.61
n=4	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(3)$
	.99	11.61	11.65	11.54	11.37	11.42	11.39	11.26	11.34
	.975	9.67	9.61	9.55	9.51	9.44	9.46	9.35	9.35
	.95	8.09	8.06	7.96	7.98	7.96	7.91	7.90	7.82
	.9	6.49	6.47	6.39	6.43	6.43	6.37	6.35	6.25
n=5	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(4)$
	.99	13.83	13.78	13.57	13.49	13.50	13.34	13.54	13.28
	.975	11.65	11.59	11.52	11.40	11.39	11.28	11.33	11.14
	.95	9.92	9.91	9.83	9.83	9.76	9.70	9.66	9.49
	.9	8.18	8.16	8.09	8.13	8.05	8.00	7.97	7.78
n=6	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(5)$
	.99	16.23	15.97	15.82	15.75	15.71	15.84	15.54	15.09
	.975	13.79	13.53	13.48	13.44	13.45	13.45	13.27	12.83
	.95	11.90	11.68	11.66	11.59	11.63	11.63	11.51	11.07
	.9	9.89	9.81	9.75	9.70	9.76	9.74	9.64	9.24
n=7	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(6)$
	.99	18.29	17.91	17.90	18.00	17.76	17.73	17.64	16.81
	.975	15.64	15.48	15.43	15.39	15.27	15.28	15.15	14.45
	.95	13.66	13.51	13.47	13.41	13.34	13.34	13.24	12.59
	.9	11.55	11.45	11.43	11.34	11.31	11.29	11.23	10.64

Table 1b: Critical values c of $W(d,n)$ such that $\Pr(W(d,n) \leq c) = p$, T=100.

	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(1)$
n=2	.99	6.57	6.46	6.46	6.53	6.58	6.56	6.57	6.63
	.975	4.98	5.00	4.88	4.99	4.97	5.02	5.01	5.02
	.95	3.87	3.84	3.77	3.84	3.83	3.86	3.84	3.84
	.9	2.74	2.70	2.67	2.72	2.71	2.73	2.71	2.71
n=3	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(2)$
	.99	9.63	9.34	9.10	9.13	9.15	9.12	9.12	9.21
	.975	7.61	7.50	7.37	7.37	7.34	7.35	7.41	7.38
	.95	6.17	6.10	5.98	5.98	6.01	5.98	6.05	5.99
	.9	4.77	4.68	4.61	4.62	4.62	4.61	4.66	4.61
n=4	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(3)$
	.99	11.71	11.52	11.52	11.25	11.44	11.26	11.35	11.34
	.975	9.62	9.48	9.54	9.34	9.44	9.35	9.38	9.35
	.95	8.07	7.96	7.94	7.84	7.88	7.81	7.82	7.82
	.9	6.44	6.39	6.36	6.31	6.34	6.27	6.28	6.25
n=5	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(4)$
	.99	13.86	13.50	13.40	13.37	13.27	13.19	13.37	13.28
	.975	11.67	11.38	11.29	11.27	11.18	11.23	11.22	11.14
	.95	9.96	9.71	9.63	9.68	9.54	9.62	9.59	9.49
	.9	8.21	7.98	7.94	7.90	7.87	7.90	7.88	7.78
n=6	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(5)$
	.99	15.84	15.36	15.30	15.32	15.24	15.21	15.32	15.09
	.975	13.60	13.09	13.08	12.98	12.99	12.97	13.07	12.83
	.95	11.70	11.33	11.33	11.28	11.24	11.21	11.29	11.07
	.9	9.75	9.53	9.48	9.46	9.42	9.40	9.39	9.24
n=7	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(6)$
	.99	17.80	17.40	17.14	17.19	17.26	17.13	17.19	16.81
	.975	15.30	15.12	14.81	14.80	14.83	14.83	14.72	14.45
	.95	13.33	13.14	12.89	12.94	12.91	12.94	12.85	12.59
	.9	11.27	11.13	10.93	10.94	10.93	10.95	10.88	10.64

Table 1c: Critical values c of $W(d,n)$ such that $\Pr(W(d,n) \leq c) = p$, T=200.

	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(1)$
n=2	.99	6.76	6.65	6.52	6.56	6.49	6.54	6.62	6.63
	.975	5.13	5.01	4.97	4.99	4.97	4.98	5.09	5.02
	.95	3.92	3.83	3.81	3.82	3.82	3.81	3.92	3.84
	.9	2.78	2.71	2.68	2.70	2.72	2.70	2.74	2.71
n=3	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(2)$
	.99	9.78	9.44	9.27	9.25	9.15	9.20	9.33	9.21
	.975	7.86	7.57	7.47	7.43	7.35	7.35	7.45	7.38
	.95	6.39	6.15	6.07	6.00	5.99	5.97	6.04	5.99
	.9	4.93	4.77	4.66	4.62	4.63	4.61	4.64	4.61
n=4	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(3)$
	.99	12.07	11.56	11.40	11.33	11.24	11.31	11.45	11.34
	.975	9.97	9.50	9.36	9.35	9.31	9.31	9.41	9.35
	.95	8.25	7.99	7.85	7.86	7.82	7.83	7.88	7.82
	.9	6.58	6.40	6.31	6.29	6.28	6.27	6.32	6.25
n=5	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(4)$
	.99	14.15	13.64	13.56	13.25	13.27	13.36	13.52	13.28
	.975	11.90	11.48	11.39	11.06	11.17	11.19	11.39	11.14
	.95	10.15	9.82	9.70	9.52	9.53	9.52	9.68	9.49
	.9	8.32	8.05	7.94	7.82	7.82	7.84	7.92	7.78
n=6	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(5)$
	.99	16.04	15.54	15.17	15.10	15.30	15.17	15.22	15.09
	.975	13.66	13.29	12.98	12.89	13.01	12.89	12.99	12.83
	.95	11.78	11.46	11.22	11.11	11.17	11.18	11.21	11.07
	.9	9.83	9.58	9.38	9.30	9.36	9.33	9.37	9.24
n=7	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(6)$
	.99	18.12	17.40	17.15	16.94	17.01	16.79	17.15	16.81
	.975	15.43	14.97	14.68	14.61	14.54	14.44	14.70	14.45
	.95	13.46	13.02	12.83	12.75	12.65	12.67	12.83	12.59
	.9	11.36	11.03	10.85	10.80	10.74	10.75	10.86	10.64

Table 1d: Critical values c of $W(d,n)$ such that $\Pr(W(d,n) \leq c) = p$, T=500.

	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(1)$
n=2	.99	6.68	6.67	6.58	6.54	6.64	6.63	6.68	6.63
	.975	4.94	5.16	4.97	5.00	5.03	5.02	5.04	5.02
	.95	3.83	3.97	3.82	3.83	3.85	3.86	3.84	3.84
	.9	2.72	2.83	2.70	2.70	2.71	2.71	2.69	2.71
n=3	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(2)$
	.99	9.33	9.10	9.20	9.15	9.31	9.24	9.24	9.21
	.975	7.58	7.46	7.52	7.36	7.52	7.41	7.62	7.38
	.95	6.07	5.98	5.88	5.98	6.08	6.03	6.20	5.99
	.9	4.65	4.56	4.56	4.59	4.65	4.63	4.74	4.61
n=4	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(3)$
	.99	11.54	11.39	11.24	11.33	11.25	11.37	11.38	11.34
	.975	9.50	9.40	9.41	9.32	9.37	9.37	9.33	9.35
	.95	8.03	7.83	7.74	7.83	7.86	7.90	7.85	7.82
	.9	6.34	6.25	6.19	6.26	6.28	6.33	6.23	6.25
n=5	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(4)$
	.99	13.43	13.18	13.13	13.26	13.20	13.27	13.25	13.28
	.975	11.30	11.07	11.11	11.20	11.11	11.17	11.12	11.14
	.95	9.63	9.47	9.35	9.59	9.49	9.53	9.45	9.49
	.9	8.04	7.84	7.74	7.85	7.80	7.81	7.88	7.78
n=6	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(5)$
	.99	15.32	15.12	15.28	15.14	15.15	15.07	15.19	15.09
	.975	13.08	12.88	12.88	12.93	12.93	12.88	12.98	12.83
	.95	11.20	11.14	11.13	11.15	11.13	11.09	11.12	11.07
	.9	9.35	9.34	9.22	9.30	9.25	9.26	9.27	9.24
n=7	p	$d=-.4$	$d=-.3$	$d=-.2$	$d=-.1$	$d=0$	$d=.1$	$d=.2$	$\chi^2(6)$
	.99	17.13	17.08	16.65	16.90	16.89	16.76	17.05	16.81
	.975	14.70	14.69	14.30	14.57	14.54	14.50	14.49	14.45
	.95	12.81	12.74	12.49	12.68	12.66	12.64	12.57	12.59
	.9	10.82	10.56	10.62	10.70	10.69	10.72	10.74	10.64

Experiment 2. The purpose of this experiment is to examine the power and the size of the test for various T and n. We consider the following 12 models:

$$
\begin{aligned}
\text{Model 1} &: & y_t &= -0.8y_{t-1} - 0.4y_{t-2} + u_t; \\
\text{Model 2} &: & y_t &= -0.9y_{t-1} - 0.4y_{t-2} + u_t; \\
\text{Model 3} &: & y_t &= -y_{t-1} - 0.4y_{t-2} + u_t; \\
\text{Model 4} &: & y_t &= 0.5y_{t-1} - 0.2y_{t-2} + 0.3y_{t-3} + 0.1y_{t-4} - 0.4y_{t-5} + u_t; \\
\text{Model 5} &: & y_t &= 0.5u_t + u_{t-1}; \\
\text{Model 6} &: & y_t &= 0.5y_{t-1} - 0.5y_{t-2} + u_t - 0.2u_{t-1}; \\
\text{Model 7} &: & y_t &= I\left(-0.3\right); \\
\text{Model 8} &: & y_t &= I\left(-0.2\right); \\
\text{Model 9} &: & y_t &= I\left(-0.1\right); \\
\text{Model 10} &: & y_t &= u_t; \\
\text{Model 11} &: & y_t &= I\left(0.1\right); \\
\text{Model 12} &: & y_t &= I\left(0.2\right).
\end{aligned}
$$

$$u_t \sim N\left(0, 1\right). \qquad t = 1, 2, ..., T.$$

Note that Model 1 to Model 6 cannot be embedded in the family of fractionally integrated processes. However, for Model 1, the first coefficient is twice the second coefficient in Model 1. Thus, for Model 1, we do not expect to reject the null hypothesis when using $n = 2$ only. For Model 2 to Model 6, which are also models under the alternative, we expect the null to be easily rejected. Model 7 to Model 12 are I(d) processes. We report the sizes of the test for these models. Table 2 reports the rejection rates of the test

$$\Pr\left(W\left(\widehat{d}, n\right) > \chi^2_\alpha\left(n-1\right)\right)$$

for $\alpha = 5\%$; $T = 50, 100, 200$; $n = 2, 3, 4, 5, 6, 7$. The number of replications is 100000.

Table 2 : $\Pr\left(W\left(\widehat{d},n\right) > \chi^2_\alpha\left(n-1\right)\right)$ for $\alpha = 5\%$.

$n\backslash T$	Model 1			Model 2			Model 3		
	50	100	200	50	100	200	50	100	200
2	.038	.035	.042	.061	.087	.143	.146	.278	.520
3	.435	.746	.966	.567	.871	.994	.725	.962	1.000
4	.528	.849	.993	.661	.943	.999	.806	.988	1.000
5	.571	.889	.997	.710	.963	1.000	.841	.993	1.000
6	.586	.906	.998	.720	.970	1.000	.848	.994	1.000
7	.602	.910	.999	.734	.972	1.000	.858	.996	1.000

$n\backslash T$	Model 4			Model 5			Model 6		
	50	100	200	50	100	200	50	100	200
2	.576	.925	.999	.709	.945	.997	.967	1.000	1.000
3	.575	.922	.998	.623	.876	.994	.930	.999	1.000
4	.581	.924	.997	.558	.849	.996	.892	.999	1.000
5	.866	.994	1.000	.535	.767	.992	.854	.993	1.000
6	.756	.978	.999	.575	.681	.987	.808	.988	1.000
7	.702	.951	.993	.593	.673	.967	.793	.989	1.000

$n\backslash T$	Model 7			Model 8			Model 9		
	50	100	200	50	100	200	50	100	200
2	.059	.043	.060	.049	.052	.050	.043	.044	.052
3	.045	.042	.050	.051	.051	.048	.059	.051	.057
4	.052	.054	.039	.056	.051	.050	.058	.041	.047
5	.055	.041	.060	.056	.051	.049	.060	.056	.044
6	.066	.047	.052	.061	.053	.049	.067	.046	.045
7	.059	.046	.044	.068	.058	.055	.057	.047	.055

$n\backslash T$	Model 10			Model 11			Model 12		
	50	100	200	50	100	200	50	100	200
2	.051	.050	.050	.055	.061	.054	.050	.049	.050
3	.050	.051	.048	.042	.051	.056	.048	.050	.049
4	.054	.051	.050	.062	.066	.054	.048	.047	.050
5	.057	.053	.052	.051	.050	.050	.050	.048	.046
6	.060	.054	.054	.059	.054	.055	.056	.049	.048
7	.067	.056	.050	.063	.050	.054	.055	.049	.049

The results are in line with our expectation. For Models 1 to 6, as the sample size becomes large, the null hypothesis will eventually be rejected. Thus, the test is consistent against a wide range of alternatives. For Models 7 to 12, the size of the test is approximately equal to 5% when sample size is large.

5. Empirical Applications

Diebold and Rudebusch (1991) and Haubrich (1993) argue that, when income follows a fractionally differenced process, the Deaton's excessive smoothness paradox can be resolved. In this section, we provide two empirical applications to examine the Deaton's paradox. All the data are obtained from the DataStream International. All variables are in constant dollars on a seasonally adjusted basis.

The first application is to test if the real disposable income per capita of the U.S. is fractionally integrated. The sample period is from 1960:Q1 to 2005:Q4 for quarterly data and from 1960 to 2005 for annual data.

We test if the real quarterly disposable income per capita and the real annual disposable income per capita are fractionally integrated[3]. The results are reported in Table 3a. From Table 3a, it is concluded that at the 5% significance level, we cannot reject the null hypothesis that the annual and quarterly real disposable income follow $I(d)$. Our second application is on the quarterly real GDP of the G7 industrial countries[4]. The sample period is from 1960:Q1 to 2005:Q4[5]. Table 3b records the values of the test statistic with $n = 2$ to 11 for the G7 countries.

In Table 3b, the estimated values of d are reported in parentheses[6]. Figures with (*) and (**) are significant at the 1 % and 5% levels respectively. Note that the estimated values of d are quite robust to the choice of n. At the 1% significance level, the null cannot be rejected for most of the G7 countries except France. The null hypothesis is rejected for France at the 5% level for all n. In general, our results suggest that most countries have a fractionally integrated GDP series.

[3] The test is performed on the drift-removed first difference of the original real disposable income data.

[4] The test is performed on the drift-removed first difference of the original real GDP data.

[5] For France, the data period is from 1963Q1 to 2005Q4.

[6] If the median estimate falls outside (-0.5,0.25), another observed estimate which falls within this range is used.

Table 3a: $W\left(\widehat{d},n\right)$ based on the first difference of the U.S. real disposable income. The estimated values of d are reported in parentheses.

Data Period	*Quaterly* 1960.1 − 2005.4	*Annual* 1960 − 2005	$\chi^2_{n-1,5\%}$
T	184	46	
$W\left(\widehat{d},2\right)$	2.31 (.154)	1.80 (.213)	3.84
$W\left(\widehat{d},3\right)$	3.10 (.156)	2.92 (.184)	5.99
$W\left(\widehat{d},4\right)$	3.77 (.156)	3.02 (.191)	7.82
$W\left(\widehat{d},5\right)$	8.94 (.156)	3.23 (.191)	9.49
$W\left(\widehat{d},6\right)$	11.01 (.156)	4.67 (.191)	11.07
$W\left(\widehat{d},7\right)$	11.06 (.156)	11.26 (.191)	12.59

6. Concluding Remarks

Inspired by the findings of Diebold and Rudebusch (1991) and Haubrich (1993) that the Deaton's (1987) paradox can be resolved by allowing the income data to be fractionally integrated, this paper develops a test which can distinguish fractionally integrated processes from other time series processes. The asymptotic distribution of the test statistic is derived. Our results provide the theoretical ground for the works of Diebold and Rudebusch (1991) and Haubrich (1993). We apply the test to the U.S. annual and quarterly per capita disposable income, and to the real GDP data of the $G7$ industrial countries. It is concluded that the U.S. real disposable income per capita is fractionally integrated. For the G7 countries, at the 5% level, we find that almost all G7 countries, except France, have a fractionally integrated GDP series.

Table 3b : $W\left(\widehat{d},n\right)$ and $\widehat{d}$

Countries	$W\left(\widehat{d},2\right)$	$W\left(\widehat{d},3\right)$	$W\left(\widehat{d},4\right)$	$W\left(\widehat{d},5\right)$	$W\left(\widehat{d},6\right)$
US	1.62 (.248)	4.53 (.248)	4.54 (.248)	6.65 (.248)	6.64 (.248)
UK	1.97 (.077)	7.59 (.077)	7.63 (.077)	7.64 (.077)	8.87 (.077)
Canada	2.00 (.157)	2.13 (.157)	8.06** (.157)	10.57** (.157)	10.83 (.157)
Japan	.01 (.108)	6.44** (.108)	7.35 (.107)	7.35 (.107)	7.77 (.107)
Italy	.29 (.169)	2.16 (.169)	5.27 (.169)	6.16 (.169)	9.23 (.169)
Germany	.684 (−.051)	.723 (−.051)	8.76** (−.051)	9.00 (−.051)	9.41 (−.051)
France	5.34** (−.239)	17.09* (−.239)	17.08* (−.238)	17.20* (−.238)	18.14* (−.237)
$\chi^2_{n-1,1\%}$	6.63	9.21	11.34	13.28	15.09
$\chi^2_{n-1,5\%}$	3.84	5.99	7.82	9.49	11.07

	$W\left(\widehat{d},7\right)$	$W\left(\widehat{d},8\right)$	$W\left(\widehat{d},9\right)$	$W\left(\widehat{d},10\right)$	$W\left(\widehat{d},11\right)$
US	7.56 (.248)	10.62 (.248)	16.08** (.248)	16.08 (.248)	16.75 (.248)
UK	9.04 (.068)	16.31 (.037)	16.50** (.033)	16.54 (.032)	16.92 (.032)
Canada	10.83 (.157)	11.34 (.157)	11.78 (.157)	11.80 (.157)	22.20** (.157)
Japan	9.76 (.107)	11.81 (.104)	11.91 (.105)	13.21 (.105)	13.21 (.105)
Italy	9.35 (.173)	9.35 (.173)	9.58 (.176)	9.60 (.176)	9.61 (.176)
Germany	9.56 (−.051)	13.26 (−.051)	13.44 (−.051)	14.43 (−.049)	14.43 (−.049)
France	18.12* (−.232)	18.12** (−.230)	18.95** (−.229)	19.02** (−.227)	21.55** (−.227)
$\chi^2_{n-1,1\%}$	16.81	18.48	20.09	21.67	23.21
$\chi^2_{n-1,5\%}$	12.59	14.07	15.51	16.92	18.31

Appendix: Proof of Theorem 1.

Note that since y_t is stationary,

$$B(n,1) - B(n,n)\Lambda(n)$$

are asymptotically multivariate normal. We have

$$B(n,1) - B(n,n)\Lambda(n) \xrightarrow{d} N(0, \Omega(d)),$$

where $\Omega(d)$ is the variance-covariance matrix of $B(n,1) - B(n,n)\Lambda(n)$. Since $\Omega(d)$ is positive definite, there exists a non-singular matrix P such that

$$\Omega(d) = PP',$$

which gives

$$\Omega(d)^{-1} = \left(P^{-1}\right)' P^{-1}$$

and

$$P^{-1}\Omega(d)\left(P^{-1}\right)' = I.$$

Define an $(n-1)$-element ψ vector as

$$\psi = P^{-1}\left(B(n,1) - B(n,n)\Lambda(n)\right).$$

The ψ variables are asymptotically multivariate normal since they are linear combinations of the $B(n,1) - B(n,n)\Lambda(n)$,

$$E(\psi) = P^{-1}E\left(B(n,1) - B(n,n)\Lambda(n)\right) = P^{-1}0 = 0,$$

$$\begin{aligned} Var(\psi) &= E\left[P^{-1}\left(B(n,1) - B(n,n)\Lambda(n)\right)\left(B(n,1) - B(n,n)\Lambda(n)\right)'\left(P^{-1}\right)'\right] \\ &= P^{-1}\Omega(d)\left(P^{-1}\right)' = I. \end{aligned}$$

Thus, ψ's are asymptotically standardized normal variables and

$$\psi'\psi \xrightarrow{d} \chi^2(n-1).$$

Now, use the fact that

$$\begin{aligned}\psi'\psi &= (B(n,1) - B(n,n)\Lambda(n))' \left(P^{-1}\right)' P^{-1} (B(n,1) - B(n,n)\Lambda(n)) \\ &= (B(n,1) - B(n,n)\Lambda(n))' \Omega(d)^{-1} (B(n,1) - B(n,n)\Lambda(n)) \\ &= W(d,n),\end{aligned}$$

that the elements in $\Omega(d)$ are continuous in d and that $\widehat{d} \xrightarrow{p} d$, we have

$$W\left(\widehat{d}, n\right) = W(d,n) + o_p(1) \xrightarrow{d} \chi^2(n-1). \blacksquare$$

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