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Multivariate Analysis

A New Method for Statistical Multidimensional Unfolding

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Consider the problem of estimating the positions of a set of targets in a multidimensional Euclidean space from distances reported by a number of observers when the observers do not know their own positions in the space. Each observer reports the distance from the observer to each target plus a random error. This statistical problem is the basic model for the various forms of what is called multidimensional unfolding in the psychometric literature. Multidimensional unfolding methodology as developed in the field of cognitive psychology is basically a statistical estimation problem where the data structure is a set of measures that are monotonic functions of Euclidean distances between a number of observers and targets in a multidimensional space. The new method presented in this article deals with estimating the target locations and the observer positions when the observations are functions of the squared distances between observers and targets observed with an additive random error in a two-dimensional space. The method provides robust estimates of the target locations in a multidimensional space for the parametric structure of the data generating model presented in the article. The method also yields estimates of the orientation of the coordinate system and the mean and variances of the observer locations. The mean and the variances are not estimated by standard unfolding methods which yield targets maps that are invariant to a rotation of the coordinate system. The data is transformed so that the nonlinearity due to the squared observer locations is removed. The sampling properties of the estimates are derived from the asymptotic variances of the additive errors of a maximum likelihood factor analysis of the sample covariance matrix of the transformed data augmented with bootstrapping. The robustness of the new method is tested using artificial data. The method is applied to a 2001 survey data set from Turkey to provide a real data example.

Keywords Bootstrapping; Least squares; Maximum likelihood factor analysis; Multidimensional unfolding; Spatial theory.

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1. Introduction

Consider the problem of estimating the positions of a set of targets in a multidimensional Euclidean space from distances reported by a number of observers when the observers do not know their own positions in the space. Each observer reports the distance from the observer to each target plus a random error. This statistical problem is the basic model for the various forms of what is called *multidimensional unfolding* in the psychometric literature. Multidimensional unfolding methodologies are based on a geometric preference model that assumes that an individual will choose the object in a multidimensional choice set that is closest to that person's ideal point in the space (Borg and Groenen, 1997, Ch. 14). The unfolding methods require that different individuals similarly perceive the objects in the choice set but they differ in their ideal points. Metric multidimensional unfolding is a statistical estimation problem where the data structure is a set of measures that are monotonic functions of Euclidean distances between a number of observers located at positions $\mathbf{x}_i$ and targets at locations $\boldsymbol{\pi}_m$. The first approach to locating the targets and the observers when the reported distances are error free is given by Schonemann (1970). A survey of the mathematics behind Schonemann's method is given by Sibson (1978).

Multidimensional unfolding is related to multidimensional scaling methods but the scaling methods developed from the approach originated by Torgerson (1952, 1958) is based on distances between the targets as reported by the observers rather than the distances to the targets. Multidimensional scaling and unfolding has been applied in marketing, anthropology, psychology, and sociology (Weller and Romney, 1990), political science (Poole, 2000; Poole and Rosenthal, 1984, 1991, 1997), and in engineering signal processing (Cahoon and Hinich, 1976).

The critical issues of the sampling properties of parameter estimates for this statistical problem have been obscured by the dominance of this literature by the work of cognitive psychologists and psychophysicists. Even in the Borg and Groenen (1997) book, which is in the Spring Series in Statistics, there is no reference to "estimation" and "parameter" in their subject index. Borg and Groenen very clearly state that the unfolding methodology was driven by the scientific imperatives of cognitive psychology.

The new method presented in this article deals with estimating the target locations and the observer positions from data that are functions of the squared distances between observers and targets observed with an additive random error in a two-dimensional space. The method builds on the work of Cahoon (1975), Cahoon et al. (1978), and Hinich (1978). The original Cahoon–Hinich (C–H) method has even been referenced in a study of strategic hospital planning by Drain and Godkin (1996).

The orientation of the coordinate system for the target map is estimable in contrast to the scaling and unfolding methods where the map is invariant to a rotation of the coordinate system. The use of scaling and unfolding in psychological applications do not use a utility based theory as does the author's method. The utility based choice theory is presented by Enelow and Hinich (1984).

The method presented in this article is a significant modification of the C–H method that yields the most accurate estimates of the target locations in a multidimensional space as well as sampling properties that could not be obtained from the original C–H approach. The C–H method could only work for a one- or two-dimensional spatial model. The new method also yields more accurate estimates of the mean and variance of the observer locations x_i than the original method.

The statistical problem will be presented in terms of squared distances between a set of observers and targets. The method can switch between a straight distance model and squared distance model. For ease of exposition, consider the squared distance model when the Euclidean space is *two*-dimensional. The method is easily extended to Euclidean spaces whose dimension is larger than two but in applications to determine the nature of political spaces a variety of methods show that the spaces are almost always either *one*- or *two-dimensional*.

2. A Statistical Quadratic Distance Model

Suppose that there are N observers and $M+1$ targets. Each observer at position $\mathbf{x}_i = (x_{i1}, x_{i2})'$ on a two-dimensional surface reports the squared Euclidean distances $S(\boldsymbol{\pi}_m, \mathbf{x}_i)$ to the targets $m = 0, 1, \ldots, M$ at locations $\boldsymbol{\pi}_m = (\pi_{m1}, \pi_{m2})'$. Each reported distance has an additive stochastic error e_{im} with mean βv_m and variance ψ_m^2. Assume that the errors e_m are independently and identically distributed and that they are independent of the observer positions $\mathbf{x}_i$.

The v_m are assumed to be known but the scale parameter β has to be estimated. Thus

$$S(\boldsymbol{\pi}_m, \mathbf{x}_i) = (\boldsymbol{\pi}_m - \mathbf{x}_i)'(\boldsymbol{\pi}_m - \mathbf{x}_i) + e_{im} = \boldsymbol{\pi}_m'\boldsymbol{\pi}_m - 2\boldsymbol{\pi}_m'\mathbf{x}_i + \mathbf{x}_i'\mathbf{x}_i + e_{im}. \tag{2.1}$$

Assuming that there are no missing distance reports, there are $(M+1)N$ observations to estimate $2(M+1+N)$ observer and target positions. There are enough observations and structure in the model to estimate all its parameters.

The ability to incorporate the bias terms βv_m in the unfolding problem is a unique feature of this method. This is important for mapping political spaces since the voting choice is usually dependent on perceived candidate quality characteristics as well as the distances of the candidates from a voter's ideal point. In this case, v_m is the mth candidate's quality score (the larger the better) and β is a negative parameter. This flexibility to incorporate bias terms into the model may prove useful in other social science applications as well as possible applications in astronomy.

The parameter estimates will be derived from the $S(\boldsymbol{\pi}_m, \mathbf{x}_i)$ using a procedure that will be presented in the next section. The estimation method is different from any of the unfolding methods discussed in Borg and Groenen (1997). Also the only assumption about the distribution of the errors e_{im} is that it has finite moments.

The C–H method reduces the dimensionality of the problem by separating the estimation of the target positions from the estimation of the observer positions. The quadratic terms $\mathbf{x}_i'\mathbf{x}_i$ are removed by subtracting the distances to one target, say target $m = 0$, from the distances to the other targets and then computing the sample $M \times M$ covariance matrix of the differences $D(\boldsymbol{\pi}_m, \mathbf{x}_i) = S(\boldsymbol{\pi}_m, \mathbf{x}_i) - S(\boldsymbol{\pi}_0, \mathbf{x}_i)$. The target whose distances are subtracted from the others is called the *reference target*. The importance of removing the quadratic terms will become clarified as the method is presented.

Since the origin of the space is not identified from the distance data and thus is arbitrary, the algebra is simplified by setting $\boldsymbol{\pi}_0 = 0$. Then

$$D(\boldsymbol{\pi}_m, \mathbf{x}_i) = \boldsymbol{\pi}_m'\boldsymbol{\pi}_m - 2\boldsymbol{\pi}_m'\mathbf{x}_i + e_{im} - e_{i0}. \tag{2.2}$$

The positions of the targets and the other parameters of the model are estimated from the sample covariance matrix of the M differences $D(\boldsymbol{\pi}_m, \mathbf{x}_i)$'s.

Assume that x_{i1} and x_{i2} are uncorrelated random variables whose variances are denoted σ_{x1}^2 and σ_{x2}^2. Then the $M \times M$ covariance matrix of the $D(\boldsymbol{\pi}_m, \mathbf{x}_i)$'s is

$$\boldsymbol{\Sigma}_D = 4\boldsymbol{\Pi}\boldsymbol{\Sigma}_x\boldsymbol{\Pi}' + \boldsymbol{\Psi} + \psi_0^2\mathbf{1} \tag{2.3}$$

where $\boldsymbol{\Pi}' = (\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_M)$ is a $2 \times M$ matrix of target positions, $\boldsymbol{\Psi}$ is an $M \times M$ diagonal matrix whose diagonal elements are the variances $\psi_m^2 = E(e_{im}^2)$ of the errors, ψ_0^2 is the variance of the error e_{i0}, $\mathbf{1}$ is an $M \times M$ matrix of ones, and $\boldsymbol{\Sigma}_x = \begin{pmatrix} \sigma_{x1}^2 & 0 \\ 0 & \sigma_{x2}^2 \end{pmatrix}$ is the 2×2 diagonal covariance matrix of the $\mathbf{x}_i = (x_{i1}, x_{i2})'$.

If the sample covariance matrix of the data was used to estimate the model then there would be third- and fourth-order moments of the unknown x_i in the expected value of the covariances. The joint density of the x_i's is unknown and for most applications the joint density will not be symmetric. Trying to estimate the third and fourth moments of the observer locations is next to impossible whereas the factorization of the sample covariance of the differences yields good results.

It is impossible to explain the method without reviewing the notation of maximum likelihood factor analysis. A maximum likelihood factor analysis based on Eq. (2.3) is presented in the next section.

3. Estimating the Target Locations

To illustrate how maximum likelihood factor analysis can be applied to (2.3), assume for a while that ψ_0 is known. Then $\boldsymbol{\Sigma}_D - \psi_0^2\mathbf{1}'\mathbf{1} = 4\boldsymbol{\Pi}\boldsymbol{\Sigma}_x\boldsymbol{\Pi}' + \boldsymbol{\Psi}$ is the $M \times M$ covariance matrix of the $\boldsymbol{\pi}_m'\boldsymbol{\pi}_m - 2\boldsymbol{\pi}_m'\mathbf{x}_i + e_{im}$. This is a standard *factor analysis* model where the factor loading matrix is the $M \times 2$ matrix $\boldsymbol{\Lambda} = 2\boldsymbol{\Pi}\boldsymbol{\Sigma}_x^{1/2}$ and $\boldsymbol{\Psi}$ is the $M \times M$ diagonal matrix of additive error variances (Lawley and Maxwell, 1971).

The unbiased sample covariance matrix of the observation vectors $\mathbf{D}_i = (D(\boldsymbol{\pi}_1, \mathbf{x}_i), \ldots, D(\boldsymbol{\pi}_M, \mathbf{x}_i))'$ is

$$\mathbf{S} = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{D}_i - \overline{\mathbf{D}})(\mathbf{D}_i - \overline{\mathbf{D}})' \tag{3.1}$$

where $\overline{\mathbf{D}} = (\overline{D}(\boldsymbol{\pi}_1), \ldots, \overline{D}(\boldsymbol{\pi}_M))'$ and $\overline{D}(\boldsymbol{\pi}_k) = \frac{1}{N}\sum_{i=1}^{N} D(\boldsymbol{\pi}_k, \mathbf{x}_i)$ is the sample mean of observations of the target k. Let $\widehat{\Lambda}(\psi_0)$ denote the maximum likelihood estimate of $\boldsymbol{\Lambda} = 2\boldsymbol{\Pi}\boldsymbol{\Sigma}_x^{1/2}$. If the observations are bounded then the asymptotic results of Anderson and Amemiya (1988) apply to the matrix $\mathbf{S} - \psi_0^2\mathbf{1}\mathbf{1}'$ and thus $\sqrt{N}[\widehat{\boldsymbol{\Lambda}}(\psi_0) - \boldsymbol{\Lambda}(\psi_0)]$ is asymptotically normal with mean zero as $N \to \infty$. Since the orientation of the two-dimensional coordinate system is not identified from the model, the estimated $M \times 2$ factor loading matrix obtained from a maximum likelihood factor analysis of $\mathbf{S} - \psi_0^2\mathbf{1}\mathbf{1}'$ is $\widehat{\boldsymbol{\Lambda}}(\psi_0) = 2\boldsymbol{\Pi}\boldsymbol{\Sigma}_x^{1/2}\mathbf{R} + \boldsymbol{\varepsilon}$ where the matrix $\mathbf{R} = \left(\begin{smallmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \sin\delta \end{smallmatrix}\right)$ is a δ angle orthogonal rotation of the coordinate system and ε is the error matrix of the estimate.

Joreskog (1967) shows that maximizing the likelihood is equivalent to minimizing the function $f(\boldsymbol{\Psi}) = \sum_{k=3}^{M}(\theta_k - \log\theta_k - 1)$ where $\theta_1 > \cdots > \theta_M$ are the ordered eigenvalues of the matrix $A(\boldsymbol{\Psi}) = \boldsymbol{\Psi}^{-1/2}\mathbf{S}\boldsymbol{\Psi}^{-1/2}$. This minimum is obtained by finding the ψ_{kk} that makes the $M - 2$ smallest eigenvalues θ_k as close as possible to one using the least squares metric. Cahoon (1975) programmed the maximum likelihood algorithm of Clarke (1970) and the present author modified the algorithm to obtain the maximum likelihood estimates $\widehat{\boldsymbol{\Lambda}}$ of $\boldsymbol{\Lambda} = 2\boldsymbol{\Pi}\boldsymbol{\Sigma}_x^{1/2}\mathbf{R}$, and the error variances $\psi_0^2, \psi_1^2, \ldots, \psi_M^2$.

This extended maximum likelihood factor analysis in the new approach to target estimation problem is implemented in a FORTRAN 95 program that we call MAP. The complicated errors-in-variables least squares estimate of ψ_0^2 based on a three-dimensional rotation used in the C–H method is now eliminated. This reduces some of the bias and variance in the parameter estimates.

The rotation angle δ, the elements of the mean ideal point $\boldsymbol{\mu}_x = E(\mathbf{x}_i)$, and the variances $\sigma_{x_1}^2$ and $\sigma_{x_2}^2$ are not identified from the structure in expression (2.3). These parameters are not estimable using any method that is only a function of the sample covariance. They are estimable using the vector of the sample means $\overline{\mathbf{D}} = (\overline{D}(\boldsymbol{\pi}_1), \ldots, \overline{D}(\boldsymbol{\pi}_M))'$, as is shown next. Once these parameters are estimated then the estimate of the target location matrix is $\widehat{\boldsymbol{\Pi}} = \frac{1}{2}\widehat{\mathbf{R}}'\widehat{\Lambda}\widehat{\boldsymbol{\Sigma}}_x^{-1/2}$ where $\widehat{\boldsymbol{\Sigma}}_x = \begin{pmatrix} \hat{\sigma}_{x1}^2 & 0 \\ 0 & \hat{\sigma}_{x2}^2 \end{pmatrix}$ is the estimated covariance matrix and $\widehat{\mathbf{R}}$ is the estimated rotation.

4. Estimating the Remaining Parameters

If the covariance matrix $\boldsymbol{\Sigma}_x$ of the observer locations $\mathbf{x}_i$ is diagonal with elements $\sigma_{x1}^2 \neq \sigma_{x2}^2$, then the north-south orientation of the axes is identified up to 180° rotations since the covariance matrix $\boldsymbol{\Sigma}_x$ is no longer diagonal if the axes are rotated. To formalize this assertion note that from expression (2.2) it follows that the expected value of each difference is $E[D(\boldsymbol{\pi}_m, \mathbf{x}_i)] = \boldsymbol{\pi}_m'\boldsymbol{\pi}_m - 2\boldsymbol{\pi}_m'\boldsymbol{\mu}_x + \beta v_m$ where $\boldsymbol{\mu}_x = E(\mathbf{x}_i)$ and v_m is the bias of the stochastic error.

Let $\mathbf{u}_m = (0, 0, \ldots, 1, 0 \cdots 0)'$ where the one is at the mth position in the M dimensional vector. Since $\boldsymbol{\Pi} = \frac{1}{2}\boldsymbol{\Sigma}_x^{-1/2}\mathbf{R}'$ because $\mathbf{R}^{-1} = \mathbf{R}'$, then $4\boldsymbol{\pi}_m'\boldsymbol{\pi}_m = \mathbf{u}_m'\boldsymbol{\Lambda}R'\boldsymbol{\Sigma}_x^{-1/2}\mathbf{R}\Lambda\mathbf{u}_m$. Thus

$$\boldsymbol{\pi}_m'\boldsymbol{\pi}_m = \frac{1}{4}\sum_{i=1}^{2}\sigma_{xi}^{-1}\left(\sum_{k=1}^{2}\lambda_{mk}r_{ik}\right)^2 \tag{4.1}$$

where λ_{mk} and r_{mk} are the mkth elements of the matrices $\boldsymbol{\Lambda}$ and $\mathbf{R}$. Similarly,

$$\boldsymbol{\pi}_m'\boldsymbol{\mu}_x = \frac{1}{2}\sum_{i=1}^{2}\sigma_{xi}^{-1}\mu_{xi}\left(\sum_{k=1}^{2}\lambda_{mk}r_{ik}\right)^2 \tag{4.2}$$

and thus it follows that

$$E[D(\boldsymbol{\pi}_m, \mathbf{x}_i)] = -\alpha_1\lambda_{m1}^2 - \alpha_2\lambda_{m2}^2 - \alpha_3\lambda_{m1}\lambda_{m2} + \alpha_4\lambda_{m1} + \alpha_5\lambda_{m2} + \beta v_m \tag{4.3}$$

where

$$\begin{gathered}\alpha_1 = \frac{\cos^2\delta}{\sigma_{x1}^2} + \frac{\sin^2\delta}{\sigma_{x2}^2} \quad \alpha_2 = \frac{\sin^2\delta}{\sigma_{x1}^2} + \frac{\cos^2\delta}{\sigma_{x2}^2} \quad \alpha_3 = \left(\frac{1}{\sigma_{x2}^2} - \frac{1}{\sigma_{x1}^2}\right)\sin(2\delta) \\ \alpha_4 = 2\left(\frac{\mu_{x1}\cos\delta}{\sigma_{x1}} + \frac{\mu_{x2}\sin\delta}{\sigma_{x2}}\right) \quad \alpha_5 = 2\left(-\frac{\mu_{x1}\sin\delta}{\sigma_{x1}} + \frac{\mu_{x2}\cos\delta}{\sigma_{x2}}\right).\end{gathered} \tag{4.4}$$

The rotation angle δ, the population mean ideal point $\boldsymbol{\mu}_x$, the variances σ_{x1}^2 and σ_{x2}^2, and the bias scale parameter β are estimated from an errors-in-variables least squares fit of the $\overline{D}(\boldsymbol{\pi}_m)$ to the estimates $\hat{\lambda}_{mk}$ of the $M \times 2$ elements λ_{mk} of the matrix $\boldsymbol{\Lambda}$ and the v_m using the plug-in model

$$\overline{D}(\boldsymbol{\pi}_m, \mathbf{x}_i) = -\alpha_1\hat{\lambda}_{m1}^2 - \alpha_2\hat{\lambda}_{m2}^2 - \alpha_3\hat{\lambda}_{m1}\hat{\lambda}_{m2} + \alpha_4\hat{\lambda}_{m1} + \alpha_5\hat{\lambda}_{m2} + \beta v_m. \tag{4.5}$$

The parameter estimates are

$$\begin{aligned}
&\hat{\delta} = \frac{1}{2}\tan^{-1}\left(\frac{\hat{\alpha}_3}{\hat{\alpha}_2 - \hat{\alpha}_1}\right) \quad \hat{\sigma}_{x1}^2 = \frac{1}{2}\left(\hat{\alpha}_1 + \hat{\alpha}_2 - \frac{\hat{\alpha}_3}{\sin(2\hat{\delta})}\right)\\
&\hat{\sigma}_{x2}^2 = \frac{1}{2}\left(\hat{\alpha}_1 + \hat{\alpha}_2 + \frac{\hat{\alpha}_3}{\sin(2\hat{\delta})}\right) \quad \hat{\mu}_{x1} = \frac{\hat{\sigma}_1}{2}(\hat{\alpha}_4\cos\hat{\delta} - \hat{\alpha}_5\sin\hat{\delta})\\
&\hat{\mu}_{x2} = \frac{\hat{\sigma}_2}{2}(\hat{\alpha}_4\sin\hat{\delta} + \hat{\alpha}_5\cos\hat{\delta}).
\end{aligned} \tag{4.6}$$

Note that that these estimates are nonlinear functions of the least squares estimates $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4, \hat{\alpha}_5)$ obtained from the errors-in-variables regression (4.5). These estimates are biased and the biases propagate through the nonlinear transformations.

The $\hat{\lambda}_{mk}$ are maximum likelihood estimates of the λ_{mk}, but the estimates in expression (4.5) are subtly biased for both finite N and asymptotically due to the errors-in-variables. The errors go to zero as N goes to infinity but the covariance matrix of the estimates is not diagonal and thus the errors on the right-hand side of (4.4) propagate in a complicated manner to the $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4, \hat{\alpha}_5)$, which then propagate to the estimates of the remaining parameters.

The parameter estimates are bootstrapped by resampling the data with replacement 100 times. The mean and standard error of each parameter estimate are computed from the 100 resampled estimates.

The simulation results presented next show that there is a bias in the target map even for large samples due to the errors-in-variables in the least square fit of (4.5). This method takes as much advantage of the statistical squared distance model as can be achieved without assuming a parametric probability model for the errors and employing a high-dimensional maximum likelihood fit.

The means and the standard errors of this method will be demonstrated next using simulations.

5. Sampling Statistics from Simulations

The present author wrote a FORTRAN 95 program to produce artificial data to feed the MAP program that implements the estimates that were just described. The results that are presented next are for a configuration of 25 targets are displayed in Fig. 1. The additive errors are a set of scaled pseudorandom independent normal variates and the observer positions are another set of pseudorandom independent normal variates. The standard deviations of the observer positions are $\sigma_{x1} = 5$ and $\sigma_{x2} = 2$. The error variances were set equal.

To keep the experiment manageable only two values of the sample size are used: $N = 200$ and $N = 200{,}000$ and the errors standard deviations were set equal to each

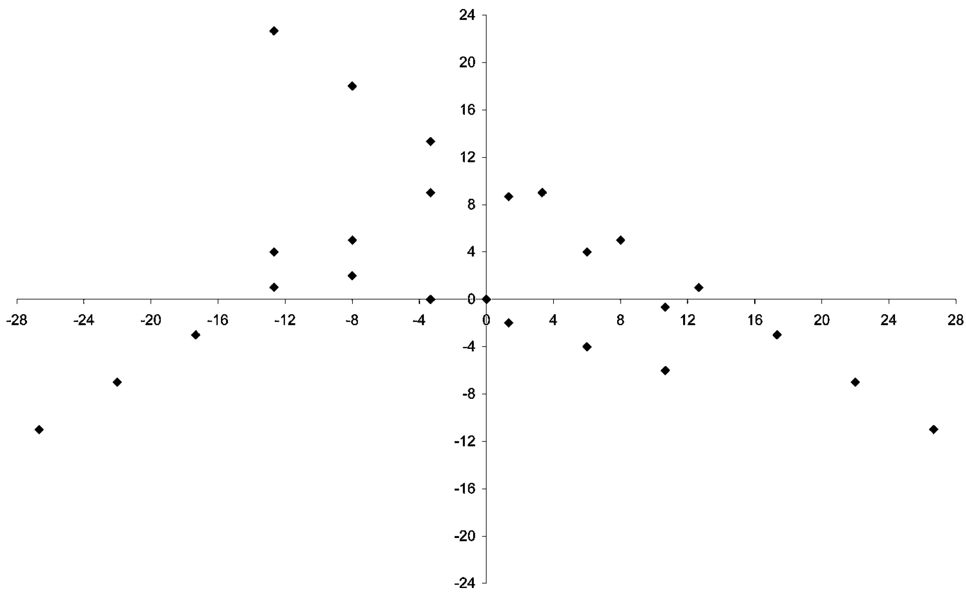

Figure 1. 25 test targets.

other. Two values of the error standard deviation $\psi = \psi_0 = \cdots = \psi_M$ were used: $\psi = 0.1$ and $\psi = 1$. The measures of goodness of fit used in this simulation are the root-mean square error (RMSE) and the maximum absolute error (MAE) of the estimated target locations with respect to the true positions in Fig. 1. The best results are obtained by using the target at the origin as the reference target.

Consider the case when the scale parameter β is zero. The results are presented in Table 1 below when the reference target is at the origin.

Since the bootstrapped standard errors of the rotation are 1.68 and 1.23 for $\psi = 0.1$ and $\psi = 1$, respectively, the differences between the RMSE values are not statistically significant at the 5% level. The error in the rotation is about 4.1 for both error variances and this error is the main source for the error in the estimation of the target positions. The estimated configuration is very close to the true target configuration.

The estimate of the mean observer position is the least robust of the errors-in-variables fits. The bootstrapped means of $\hat{\mu}_{x1}$ and $\hat{\mu}_{x2}$ are very similar to the true estimates. The bootstrapped standard errors are 0.07 for $\hat{\mu}_{x1}$ and 0.06 for $\hat{\mu}_{x2}$ for

Table 1
Reference target at the origin

N	ψ	RMSE	MAE	$\hat{\mu}_{x1} - \mu_{x1}$	$\hat{\mu}_{x2} - \mu_{x2}$	$\hat{\sigma}_{x1} - \sigma_{x1}$	$\hat{\sigma}_{x2} - \sigma_{x2}$
2.e2	0.1	2.85	14.27	−0.26	0.25	0.05	−0.15
2.e2	1.0	2.74	13.72	−0.25	0.24	0.05	−0.15
2.e5	0.1	2.70	13.48	−0.08	0.05	0.01	0.00
2.e5	1.0	2.70	13.48	−0.08	0.05	0.01	0.00

both error variances. Thus, the errors of $\hat{\mu}_{x1}$ and $\hat{\mu}_{x2}$ are not statistically significant for both error variances at the 0.1% level.

For $N = 200{,}000$ and both error variances the bootstrapped standard errors are 0.008 for $\hat{\mu}_{x1}$ and 0.003 for $\hat{\mu}_{x2}$. Thus the errors are statistically significant at the 0.1% level. The larger sample size improved the accuracy of the estimate of the mean ideal point but the bias is more statistically significant using the bootstrap standard errors.

For $N = 200$ the bootstrapped standard errors of the ideal point standard deviations $\hat{\sigma}_{x1}$ and $\hat{\sigma}_{x2}$ are 0.27 and 0.09 for both error variances. Thus the errors are not statistically significant at the 5% level and similarly for the $N = 200{,}000$ run. The bootstrapped standard errors are 0.008 and 0.003 for both error variances for the larger sample size. Thus the biases in the standard deviation estimates are insignificant for this simulation.

Now consider the results when $\beta = 1$ and the v_m values of the errors have a pseudorandom normal distribution with a zero mean and unit variance. For $N =$ 200, the RMSE is 2.91 for $\psi = 0.1$ and 2.85 for $\psi = 1$. The maximum absolute errors are 14.56 and 14.24, respectively. These results are statistically the same as when $\beta = 0$. The same holds for the other parameter estimates for all the cases, which is not surprising since the addition of one independent variable that is almost uncorrelated with the other five variables will not significantly change the other five estimates.

The error for the estimate of the scale parameter for $N = 200$ is $\hat{\beta} - \beta = 0.18$ for $\psi = 0.1$ and $\hat{\beta} - \beta = 0.06$ for $\psi = 1$. The bootstrapped standard errors are 0.01 and 0.12, respectively. The error in $\hat{\beta}$ for $\psi = 1$ is not statistically significant at the 5% level but it is statistically significant at the 0.1% for $\psi = 0.1$.

The errors become much larger if the reference target is not near the origin. To obtain good results the user has to know which target is closest to the origin or the user has to try different references and determine which is best in terms of the user's prior beliefs about the target configuration. In the Turkish politics example given in the final section we believed that the party ANAP was closest to the origin since it was the pivot party in the ruling coalition for many years.

If the reference target is the point, $T_a = (11.55, -4.01)$ in Fig. 1 and $\beta = 0$. Then the RMSE is 34.8 for $N = 200$ and $\psi = 0.1$ rather than 2.85 for the origin reference target. The maximum absolute error is 174.0. The errors of the estimates of the coordinates of the mean observer positions are $\hat{\mu}_{x1} - \mu_{x1} = 4.5$ and $\hat{\mu}_{x2} - \mu_{x2} = 5.72$ for $\psi = 0.1$, which are more than ten times the errors when the origin is the reference target. The reason for the larger differences between the true positions of the targets and their estimates is the increased error in the least squares estimates of α_4 and α_5. The estimates for the zero reference are $\hat{\alpha}_4 = 0.88$ and $\hat{\alpha}_5 = 0.71$, whereas the estimates for the T_a reference are $\hat{\alpha}_4 = 4.57$ and $\hat{\alpha}_5 = 5.65$. The first three estimates are the same for both reference targets.

When the reference target is $T_b = (-0.19, 8.77)$, the errors of the estimates of the coordinates of the mean observer positions are $\hat{\mu}_{x1} - \mu_{x1} = 1.79$ and $\hat{\mu}_{x2} - \mu_{x2} =$ 8.85 for $\psi = 0.1$. The RMSE is 27.77 and the maximum absolute error is 138.86. The estimates for fourth and fifth least squares estimates are $\hat{\alpha}_4 = -0.33$ and $\hat{\alpha}_5 = 11.2$. The fourth estimate has the wrong sign and the fifth is much larger than the estimate when the origin is the reference. There is no way to eliminate the errors since these parameters are identified using the equations in (4.5) and both the independent and dependent variables have stochastic error components. Yet keep in mind that standard unfolding methods do not estimate the rotation of the configuration nor the mean and standard deviations of the observer positions.

6. Estimating the Observer Locations

The ith observer location is estimated by a least squares fit of the linear system

$$\overline{D}(\boldsymbol{\pi}_m) = \hat{\boldsymbol{\pi}}_m' \hat{\boldsymbol{\pi}}_m - 2\hat{\boldsymbol{\pi}}_m' \mathbf{x}_i \tag{6.1}$$

which is a sample version of expression (2.2). This fit is also an errors-in-variables least squares and so the estimates are biased. To test the accuracy of the estimation, the simulation program calculated the percent of the true observer locations $\mathbf{x}_i$ that are closest to the true position of target $\boldsymbol{\pi}_m$ for each m. The percent of the estimated observer locations $\hat{\mathbf{x}}_i$, that are closest to each target is also computed. For $N = 200$ and $\psi = 0.1$, the 28% of the true $\mathbf{x}_i$'s were closer to the true location of target 22. This was the largest percentage for the run. For the estimated configuration, 20% of the estimated observer positions were closest to target 22 yielding an 8% error. The second largest true percentage was 16.5% for target 21 and the percentage for the estimates was 9% yielding a 7.5% error. The target at the origin, the reference target, had a true percentage of 15% and so did the estimates. The errors for the other percentages were smaller than 7.5%. The results for $\psi = 1$ were surprisingly slightly better than for the smaller error variance.

The differences for $N = 200{,}000$ were smaller. For $\psi = 1$ the percentage for the target with the largest true percentage was 22.5% yielding a 5.5% error. The percentage for the reference target was 19.7% yielding an error of 4.2%. For $\psi = 0.1$ the maximum error was 3.5% and the rest were about 1%.

7. A Turkish Politics Example

The MAP method is used to estimate the positions of the respondents (observers) and the candidates or parties (targets) in a latent political space depending on the political system of a democracy using a set of survey questions.

The application of the method to the spatial theory of politics requires a set of assumptions relating the data to a spatial model. First, the scores given to each party is assumed to be a monotonically decreasing function of the Euclidian distance between the position of the party in the space and the most preferred ideological position of the respondent. This position is called the ideal point. The respondent is not required to articulate that position but rather it is an unobserved position in the latent space. Second, the constellation of the party positions in the latent space is assumed to be the same across all respondents. Only the personal ideal points differ across respondents. The method is then applied to these scores that we get from the respondents.

The method was applied to a data set from a public opinion survey taken in 2001. For a complete description of the survey and the analysis see Çarkoğlu and Hinich (2003). The data comes from a nation-wide representative survey of urban population conducted during the chaotic weeks of the second economic crisis of February 2001. A total of 1201 face-to-face interviews were conducted in 12 of the 81 provinces. The questionnaires were administered between February 20 and March 16 2001 using a "random sampling" method with an objective to represent the nation-wide voting age urban population living within municipality borders, in which the urban population figures of 1997 census data were taken as the basis.

Each respondent was asked to grade the seven major parties in terms of how well that party would impact on the respondent's family if the party were to receive a majority of the seats in the parliament. These parties obtained 94.8% of the urban vote in 1999 elections. However, as of February–March 2001 these parties comprised only the preferences of 42.3% of this sample. Similar to opinion poll results reported in the media, our findings also indicate that while 6% of the respondents report that they will not cast their vote and about 5% were undecided as to which party to vote for. More significantly, nearly 33% of the respondents indicated that they will not cast their vote for any one of the existing parties. Given the continual crisis atmosphere in the country, the erosion of electoral support for the coalition partners this is not surprising. Among the opposition only the left leaning CHP party and pro-Kurdish HADEP party maintained their urban constituencies. The rest of the opposition parties had lost their supporters.

The results of a two-dimensional latent ideological spatial map of these parties together with the estimated respondents' ideal points are presented in the Fig. 2. The horizontal axis appears to posit the pro-Islamist FP in one extreme as opposed to the secularist left leaning CHP. The relative positions of the rest of the parties fit our expectations about the religious cleavage in Turkish politics. The nationalist MHP turns out to be the closest one to the position of the pro-Islamist FP on this axis. Among the centrist parties DYP is slightly closer to the pro-Islamist end. DSP and CHP are clustered together on the opposing end of this dimension placed to

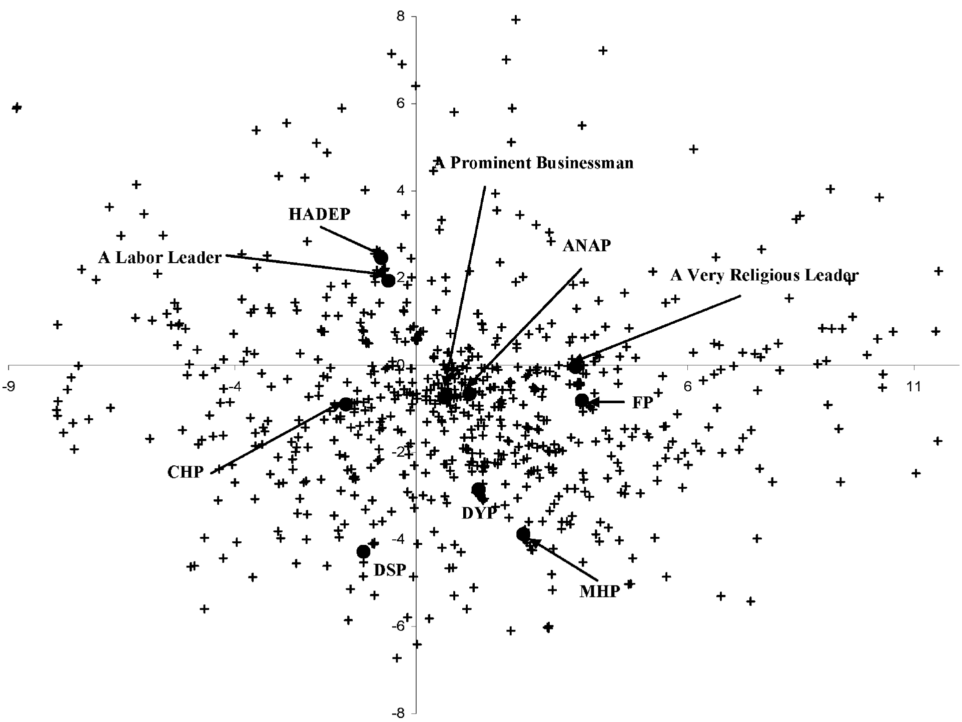

Figure 2. Estimated ideal points and party positions, full sample.

the left of ANAP's centrist position. It is noticeable that HADEP's position on this dimension is closer in the perceptions of our respondents to the secularist left of DSP and CHP.

The vertical axis has the Kurdish HADEP on one extreme and the nationalist MHP and DSP on the other. While ANAP, CHP, and FP's positions come close to the center on this dimension, DYP is placed closer to the nationalist MHP and DSP's opposing end. It has been suggested that FP's strong showing in the East and Southeastern provinces where the bulk of Kurdish population lives is evidence of FP's appeal to the Kurdish electorate. Similarly, the religiously conservative Kurdish constituency was seen by many Turkish politics scholars as a cause for ideological closeness of HADEP and FP. The spatial map shows that in the perceptions of the urban population, HADEP is nowhere close to the position of FP on the two-dimensional political space estimated from the data using MAP.

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